ON A CONJECTURE OF PAPPAS AND RAPOPORT

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Abstract. We prove a conjecture of Pappas and Rapoport about the existence of "canonical" integral models of Shimura varieties of Hodge type with quasi-parahoric level structure at a prime p. For these integral models, we moreover show uniformization of isogeny classes by integral local Shimura varieties and prove a conjecture of Kisin and Pappas on local model diagrams.

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1. Introduction

1.1. **Background.** Fix a prime p. Pappas and Rapoport have recently determined conditions which uniquely characterize p-adic integral models of Shimura varieties with parahoric level at p [PR24]. Integral models satisfying these conditions are called *canonical integral models*, and Pappas and Rapoport have conjectured the existence of such models in general. Moreover, they prove the conjecture for Shimura varieties of Hodge type, under the assumption that the level subgroup K_p at p is a stabilizer parahoric (or connected parahoric). In this article, we prove the existence of canonical integral models for Hodge-type Shimura varieties with arbitrary parahoric level at p and, more generally, with quasi-parahoric level at p.

When the level subgroup at p is hyperspecial, a collection of smooth integral models for a given Shimura variety can be uniquely characterized by an extension property, similar to the Néron mapping property, see [Mil92], [Moo98]. In this case Kisin has constructed smooth integral models satisfying the extension property for Shimura varieties of abelian type [Kis10]. In this article, we are most interested in the case where the level subgroup at p is (more generally) parahoric in the sense of [BT84]. In such cases, even the most accessible Shimura varieties (for example, the Siegel modular varieties) have integral models with complicated singularities, see e.g., [Rap05], and such models are not so easily characterized.

The key innovation of Pappas and Rapoport in [PR24], building on earlier work of Pappas (see [Pap23]), was that integral models of Shimura varieties can be characterized by the existence of a universal p-adic shtuka (in the sense of [SW20]) which satisfies certain compatibilities. In this article we work in reverse, in a sense. We take as a starting point the notion that a shtuka should exist over some integral model of the given Shimura variety at (quasi-)parahoric level, and that such a shtuka should be compatible with transition morphisms between varying levels. Following these ideas, we first define a v-sheaf supporting a universal shtuka, which we then show is the v-sheaf associated to an integral model of the given Shimura variety at parahoric level. We explain our results and methods in more detail below.

1.2. **Main Results.** Let (G, X) be a Shimura datum with reflex field E. Let p be a prime number, let v be a prime of E above p and let E be the v-adic completion of E with ring of integers $\mathcal{O}_E$ and residue field k_E. We write $G = \mathsf{G} \otimes \mathbb{Q}_p$ and let $K_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup. For $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ a neat compact open subgroup, we write $K = K^p K_p$. We denote by $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})/\operatorname{Spec}(E)$ the base change to E of the canonical model of the Shimura variety at level K over $\operatorname{Spec}(\mathsf{E})$.

We will consider systems $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$ of normal schemes $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$, flat, of finite type, and separated over $\mathcal{O}_E$, with generic fibers $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$; here K^p runs over all neat compact open subgroups of $\mathsf{G}(\mathbb{A}_f^p)$. Pappas and Rapoport give axioms for such systems, see [PR24, Conjecture 4.2.2], and show that systems satisfying their axioms are unique if they exist, see [PR24, Theorem 4.2.4]. They conjecture that systems satisfying their axioms always exist, see [PR24, Conjecture 4.2.2].

The conjecture of Pappas and Rapoport is known when G is a torus, see [Dan22], and, under the assumption that $p > 2$, when K_p is hyperspecial and (G, X) is of abelian type, see [IKY23]. It is known moreover when (G, X) is of Hodge type and K_p is a *stabilizer parahoric*; a parahoric subgroup whose associated smooth affine group scheme over $\mathbb{Z}_p$ is a stabilizer of a point in the extended building of G, see [PR24, Theorem 4.5.2]. Our main theorem extends this result to all parahoric subgroups.[1]

Theorem I (Theorem 4.2.3)**.** *If* (G, X) *is of Hodge type, then there exists a system* $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$ *satisfying* [PR24, Conjecture 4.2.2].

Theorem I is used in work of one of us (PD) and Youcis, see [DY24], to prove [PR24, Conjecture 4.2.2] for almost all (and all if $p \geq 5$) Shimura varieties of abelian type. Without Theorem I, the results of *loc. cit.* would have strong restrictions for Shimura varieties of type $D^{\mathbb{H}}$ in the sense of [Mil05, Appendix B].

We remark that recent work of Takaya [Tak24] also proves [PR24, Conjecture 4.2.2] under the more restrictive assumption that K_p is contained in a hyperspecial subgroup K_p' of $G(\mathbb{Q}_p)$, assuming the conjecture holds for K_p'. Such a K_p is necessarily a stabilizer parahoric, so in the Hodge-type case the result of Takaya follows from the work of Pappas and Rapoport. The results of Takaya therefore do not intersect

[1] In Theorem 4.2.3, we prove an extension of [PR24, Conjecture 4.2.2] to quasi-parahoric subgroups.

with ours. We mention also that the methods of Takaya, while similar in spirit to ours, crucially require smoothness and so they do not apply in our situation.

$$\begin{array}{ccccc} & & \widetilde{\mathscr{S}_K}(\mathsf{G},\mathsf{X}) & & \\ & \overset{\pi}{\swarrow} & & \overset{q}{\searrow} & \\ \mathscr{S}_K(\mathsf{G},\mathsf{X}) & & & & \mathbb{M}_{\mathcal{G},\mu}, \end{array} \tag{1.2.1}$$

where π is a $\mathcal{G}$-torsor and q is a smooth, $\mathcal{G}$-equivariant morphism. Here $\mathbb{M}_{\mathcal{G},\mu}$ is the local model associated to $\mathcal{G}$ and μ, see e.g., [AGLR22]. If a diagram as in (1.2.1) exists, then the singularities of $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ are (étale-locally) modeled by those of the (often simpler) scheme $\mathbb{M}_{\mathcal{G},\mu}$. When $\mathcal{G}$ is a stabilizer parahoric and p is coprime to $2 \cdot \pi_1(G^{\mathrm{der}})$, the existence of a diagram (1.2.1) is shown in [KP18] assuming that G splits over a tamely ramified extension, and in [KZ21], under the (weaker) assumptions that G is acceptable and R-smooth (see [KZ21, Definition 3.3.2] and [DY24, Definition 2.10]).

In [PR24, Section 4.9.1], Pappas and Rapoport construct an analogous diagram at the level of v-sheaves for any integral model $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ which admits a $\mathcal{G}$-shtuka. A diagram as in (1.2.1) which recovers the Pappas-Rapoport v-sheaf diagram is called a *scheme-theoretic local model diagram*, [PR24, Definition 4.9.1]. Pappas and Rapoport conjecture the existence of scheme-theoretic local model diagrams in general, see [PR24, Conjecture 4.9.2]. When $\mathcal{G}$ is a stabilizer (quasi-)parahoric, the existence of a scheme-theoretic local model diagram follows from results of [KP18] and [KZ21]; we explain this in detail in Appendix A. In our second main theorem, we prove the conjecture of Pappas and Rapoport (and therefore the conjecture of Kisin and Pappas) for many Shimura varieties of Hodge type when $\mathcal{G}$ is an arbitrary parahoric.

Theorem II (Theorem 4.3.6)**.** *Let (G,X) be a Hodge type Shimura datum, and let $K_p = \mathcal{G}(\mathbb{Z}_p)$ be a parahoric subgroup of $\mathsf{G}(\mathbb{Q}_p)$. If p is coprime to $2 \cdot \pi_1(G^{\mathrm{der}})$ and the group G is acceptable and R-smooth, then $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ admits a scheme-theoretic local model diagram.*

Other recent advances in the theory of integral models of Shimura varieties of Hodge type that require one to restrict to stabilizer parahorics are Rapoport–Zink uniformization of isogeny classes and the existence of CM lifts, see [Zho20, Theorem 1.1], [vanH20, Theorem I,II], [GLX23, Corollary 1.4, Corollary 6.3]. We show that the proof of [GLX23, Corollary 6.3] can be combined with Theorem I to prove uniformization in full generality, see Corollary 4.4.3. We expect that Corollary 4.4.3 can be used to prove the existence of CM lifts of isogeny classes when G is quasi-split.

1.3. **A sketch of the proof of Theorem I.** From now on, we change our notation and let $\mathcal{G}^\circ$ be a parahoric group scheme which is the relative identity component of a stabilizer Bruhat–Tits group scheme $\mathcal{G}$, see Section 2.2. Fixing $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$, we will write $K_p^\circ = \mathcal{G}^\circ(\mathbb{Z}_p)$, $K_p = \mathcal{G}(\mathbb{Z}_p)$, and $K^\circ = K^p K_p^\circ$.

As usual, to construct integral models of Shimura varieties of level K and K°, we choose a Hodge embedding $(\mathsf{G},\mathsf{X}) \to (\mathsf{G}_V,\mathsf{H}_V)$, where $\mathsf{G}_V = \mathrm{GSp}(V,\psi)$ for a symplectic space (V, ψ) over $\mathbb{Q}$. We then choose a lattice $\Lambda \subset V \otimes \mathbb{Q}_p$ such that $\mathcal{G}$ is the stabilizer of Λ, and define $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$ as the normalization of the Zariski closure of $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$ in an integral model of the Shimura variety for $(\mathsf{G}_V, \mathsf{H}_V)$ at level K_Λ. The arguments in [PR24] show that $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$ satisfies (a generalization to quasi-parahoric subgroups of) the axioms of [PR24, Conjecture 4.2.2]. We define $\mathscr{S}_{K^\circ}(\mathsf{G}, \mathsf{X})$ to be the normalization of $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$ in $\mathbf{Sh}_{K^\circ}(\mathsf{G}, \mathsf{X})$.

1.3.1. The most important part of the axioms of [PR24, Conjecture 4.2.2] is the existence of a $\mathcal{G}$-shtuka on $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$, encoded as a morphism of v-stacks (here the notation $(-)^{\diamond/}$ denotes a variant of the v-sheaf associated to a $\mathbb{Z}_p$-scheme, see Section 2.1.5 below, and μ is the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of G coming from the Hodge cocharacter and the place v)

$$\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/} \to \mathrm{Sht}_{\mathcal{G},\mu}.$$

Yet it is not a priori clear that there is a $\mathcal{G}^\circ$-shtuka on $\mathscr{S}_{K^\circ}(\mathsf{G}, \mathsf{X})$, or in other words, that there is a dotted arrow making the following diagram commutative

$$\begin{array}{ccc} \mathscr{S}_{K^\circ}(\mathsf{G}, \mathsf{X})^{\diamond/} & \dashrightarrow & \mathrm{Sht}_{\mathcal{G}^\circ,\mu} \\ \downarrow & & \downarrow \\ \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/} & \longrightarrow & \mathrm{Sht}_{\mathcal{G},\mu}. \end{array} \tag{1.3.1}$$

In fact, considerations from our companion paper [DvanHKZ24] lead us to believe that such a diagram exists and is cartesian. A computation of [KP18, Section 4.3] suggests that the left vertical map in the diagram should be finite étale.

The rough strategy of the proof now goes as follows: We show that $\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/} \to \mathrm{Sht}_{\mathcal{G},\mu}$ and $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu}$ factor through an open and closed substack $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1} \subset \mathrm{Sht}_{\mathcal{G},\mu}$. We then show that $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ is an étale torsor under a finite abelian group Λ, see Theorem III and Corollary IV below. By pulling back this cover along the map $\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$, we get an étale Λ-torsor $\mathscr{Y} \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/}$. We show, using a result of one of us (DK) [Kim24], that $\mathscr{Y}$ must be isomorphic to $\mathscr{S}_{K^\circ}(\mathsf{G}, \mathsf{X})^{\diamond/}$ over $\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/}$, see Proposition 2.3.1. This then implies that $\mathscr{S}_{K^\circ}(\mathsf{G}, \mathsf{X})^{\diamond/} \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond/}$ is an étale Λ-torsor and establishes the existence of the dotted arrow in (1.3.1). The resulting diagram is Cartesian, and the rest of the proof of Theorem I is now routine.

1.3.2. *Moduli stacks of quasi-parahoric shtukas.* The stack of $\mathcal{G}$-shtukas $\mathrm{Sht}_{\mathcal{G},\mu}$ is not as well behaved as the stack of $\mathcal{G}^\circ$-shtukas $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$. For example the image of

$$\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Bun}_G$$

is given by the open substack corresponding to the μ^{-1}-admissible elements $B(G, \mu^{-1}) \subset B(G) = |\mathrm{Bun}_G|$, see Lemma 3.1.9; the analogous statement generally fails for $\mathrm{Sht}_{\mathcal{G},\mu}$.

Our first order of business is to show that this failure can be rectified by restricting to the preimage $\mathrm{Sht}_{\mathcal{G},\mu}^{\kappa=-\mu^\natural}$ of $-\mu^\natural$ under the Kottwitz map (see [FS21, Theorem

III.2.7])

$$|\mathrm{Sht}_{\mathcal{G},\mu}| \to |\mathrm{Bun}_G| \to \pi_1(G)_{\Gamma_p},$$

see Proposition 3.1.10. Following ideas of [PR22, Section 4], we show the following (see Section 3.3 for the notation).

Theorem III (Theorem 3.3.5)**.** *There is a finite decomposition*

$$\coprod_{\delta\in\Pi_{\mathcal{G}}} [\mathrm{Sht}_{\mathcal{G}^\circ_\delta,\mu,\mathcal{O}_{\breve{E}}}/\pi_0(\mathcal{G}_\delta)^\phi] \simeq \mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}.$$

For $\delta = 1 \in \Pi_{\mathcal{G}}$, we have $\mathcal{G}^\circ_\delta = \mathcal{G}^\circ$. In particular, this establishes the following corollary, which clarifies the relationship between the stack of $\mathcal{G}^\circ$-shtukas with one leg bounded by μ with that of $\mathcal{G}$-shtukas. We see that the image of $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu}$ defines an open and closed substack $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1} \subset \mathrm{Sht}_{\mathcal{G},\mu}$.

Corollary IV (Corollary 3.3.7)**.** *The morphism* $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ *is a torsor for the abelian group* $\pi_0(\mathcal{G}_\delta)^\phi$*.*

As explained above, Corollary IV allows us to prove that $\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})^{\diamondsuit/}$ is the fiber product of $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\diamondsuit/}$ with $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$ over $\mathrm{Sht}_{\mathcal{G},\mu}$.

1.3.3. Let us close the introduction with some comments on the proof of Theorem II. Theorem II contains two separate assertions: That there exists a diagram as in (1.2.1) for $\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})$, and that the diagram recovers the one of [PR24, Section 4.9.1] at the level of v-sheaves. Our strategy is to verify both assertions for $\mathscr{S}_K(\mathsf{G},\mathsf{X})$, and then deduce the two simultaneously for $\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})$.

Under the assumptions in Theorem II, the existence of a diagram (1.2.1) is proved for $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ in [KZ21]. Pappas and Rapoport point out that the construction of *loc. cit.* provides a scheme-theoretic local model diagram in [PR24, Section 4.9.2]. We verify this statement in Appendix A.

Given a scheme-theoretic local model diagram for $\mathscr{S}_K(\mathsf{G},\mathsf{X})$, we obtain in particular a $\mathcal{G}$-torsor on $\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})$ by pullback, and we have to show this torsor admits a reduction of structure group to $\mathcal{G}^\circ$. Such a reduction exists at the level of v-sheaves by functoriality of the construction, so the crux of the argument is to show that this arises from a reduction at the level of schemes. This is done in Proposition 4.3.3.

1.4. **Outline of the paper.** In Section 2 we recall preliminaries on perfectoid geometry and Bruhat–Tits theory, and we prove a key technical result, Proposition 2.3.1. In Section 3 we study the moduli stack of $\mathcal{G}$-shtukas for a quasi-parahoric group $\mathcal{G}$ and its relationship to the moduli stack of $\mathcal{G}^\circ$-shtukas for the parahoric group scheme $\mathcal{G}^\circ$ associated with $\mathcal{G}$. This culminates in the proof of Theorem III. Finally, in Section 4, we recall the conjecture of Pappas and Rapoport, and prove our main result, Theorem I. We close by proving Theorem II, and proving Rapoport–Zink uniformization, see Theorem 4.4.1. In Appendix A, we verify that the local model diagrams of [KP18, KZ21] give scheme-theoretic local model diagrams in the sense of [PR24, Conjecture 4.9.2], for stabilizer Bruhat–Tits group schemes.

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2. Preliminaries

2.1. **Recollections from** [FS21]**.** We begin by establishing notation and recalling some definitions from the theory of v-sheaves. For a more comprehensive background, we refer the reader to [SW20], [FS21], and [PR24, Section 2.1]. Throughout this section, we let k be a perfect field of characteristic p, and write Perf_k for the category of *affinoid* perfectoid spaces over k. If $k = \mathbb{F}_p$ we will write $\mathrm{Perf} = \mathrm{Perf}_{\mathbb{F}_p}$.[2]

For any perfectoid space S over $\mathbb{F}_p$, we write $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ for the analytic adic space defined in [SW20, Proposition 11.2.1]. In particular, when $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ is given by

$$S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) = \mathrm{Spa}(W(R^+)) \setminus \{[\varpi] = 0\},$$

where $[\varpi]$ denotes the Teichmüller lift to $W(R^+)$ of a fixed pseudouniformizer ϖ in R^+, and where $W(R^+)$ denotes the p-typical Witt vectors of R^+. The Frobenius for $W(R^+)$ restricts to a Frobenius operator Frob_S on $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$. By [SW20, Proposition 11.3.1], any untilt $S^\sharp$ of S determines a closed Cartier divisor $S^\sharp \hookrightarrow S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$.

For S in Perf, define $Y_S = S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) \setminus \{p = 0\}$. If $S = \mathrm{Spa}(R, R^+)$ we write also $Y(R, R^+)$ for Y_S. For any $S = \mathrm{Spa}(R, R^+)$ in Perf, one defines a function (here $|X|$ denotes the underlying topological space of an adic spaces or v-sheaf)

$$\kappa : |S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)| \to [0, \infty)$$

by $\kappa(x) = (\log|[\varpi](\tilde{x})|)/(\log(|p(\tilde{x})|)$, where $\tilde{x}$ denotes the maximal generalization of $x \in |S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)|$, see [FS21, Proposition II.1.16] for details. For any interval $I = [a, b] \subset [0, \infty)$ with rational endpoints, denote by $\mathcal{Y}_I(S)$ the open subset of $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ given by

$$\mathcal{Y}_I(S) = \{|p|^b \le |[\varpi]| \le |p|^a\} \subset \kappa^{-1}(I).$$

One extends this definition to open intervals in the obvious way. In particular, we have $\mathcal{Y}_{[0,\infty)}(S) = S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ and $\mathcal{Y}_{(0,\infty)}(S) = Y_S$.

2.1.1. By [FS21, Proposition II.1.16], for any S in Perf the action of Frob_S on Y_S is free and totally discontinuous. We therefore obtain a well-defined quotient

$$X_S = Y_S / \mathrm{Frob}_S^{\mathbb{Z}},$$

which is the *relative adic Fargues–Fontaine curve over* S.

Let G be a reductive group over $\mathbb{Q}_p$. Following [FS21], we denote by $\mathrm{Bun}_G(S)$ the groupoid of G-torsors on X_S. By [FS21, Theorem III.0.2], the presheaf of groupoids Bun_G on Perf sending S to $\mathrm{Bun}_G(S)$ is a small v-stack.

[2] In the literature, Perf_k usually denotes the category of *all* perfectoid spaces over k.

2.1.2. For a choice of algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ we set $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$ and $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p]$. Let σ be the automorphism of $\breve{\mathbb{Q}}_p$ induced by the absolute Frobenius on $\overline{\mathbb{F}}_p$. Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$, equipped with the topology coming from the *opposite* of the partial order defined in [RR96, Section 2.3]. The formation of $B(G)$ is invariant under extensions of algebraically closed fields $\overline{\mathbb{F}}_p \hookrightarrow F$. Indeed, for such an extension, the natural map $G(\breve{\mathbb{Q}}_p) \hookrightarrow G(W(F)[1/p])$ induces a bijection on σ-conjugacy classes.

By [Vie21, Theorem 1], there is a homeomorphism

$$|\mathrm{Bun}_G| \xrightarrow{\sim} B(G).$$

If μ is a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of minuscule cocharacters, we let $B(G, \mu^{-1}) \subset B(G)$ be the (open) subset of μ^{-1}-admissible elements, as defined in [KMPS22, Section 1.1.5]; this defines an open substack

$$\mathrm{Bun}_{G,\mu^{-1}} \subset \mathrm{Bun}_G$$

via [Sch17, Proposition 12.9]. Explicitly, for S in Perf, $\mathrm{Bun}_{G,\mu^{-1}}(S)$ consists of maps $S \to \mathrm{Bun}_G$ for which the induced map on topological spaces factors through $B(G, \mu^{-1}) \subset B(G) \cong |\mathrm{Bun}_G|$.

2.1.3. For $b \in G(\breve{\mathbb{Q}}_p)$, we let $[b] \in B(G)$ denote the σ-conjugacy class of b. We recall the following result.

Theorem 2.1.4. [FS21, Theorem III.0.2] *The subfunctor*

$$\mathrm{Bun}_G^{[b]} = \mathrm{Bun}_G \times_{|\mathrm{Bun}_G|} \{[b]\} \subseteq \mathrm{Bun}_G$$

is locally closed. Moreover its base change to $\mathrm{Spd}(\overline{\mathbb{F}}_p)$ *is isomorphic to* $[\mathrm{Spd}(\overline{\mathbb{F}}_p)/\tilde{G}_b]$, *where* $\tilde{G}_b = \mathrm{Aut}(\mathcal{E}_b)$ *and* $\mathcal{E}_b \in \mathrm{Bun}_G(\mathrm{Spd}(\overline{\mathbb{F}}_p))$ *corresponds to* b *(see* [Ans23, Theorem 5.3]*).*

For any v-stack $\mathscr{Y}$ on Perf equipped with a morphism $\mathscr{Y} \to \mathrm{Bun}_G$, we write

$$\mathscr{Y}^{[b]} := \mathscr{Y} \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{[b]}. \tag{2.1.1}$$

2.1.5. If X is a pre-adic space over $\mathrm{Spa}(\mathbb{Z}_p)$ in the sense of [SW20, Section 3.4], we let $X^\lozenge$ denote the set-valued functor on Perf given by

$$X^\lozenge(S) = \{(S^\sharp, f)\}/\text{isom.}$$

for any S in Perf, where $S^\sharp$ is an untilt of S and $f : S^\sharp \to X$ is a morphism of pre-adic spaces. This determines a v-sheaf on Perf by [SW20, Lemma 18.1.1]. For a Huber pair (A, A^+) we write $\mathrm{Spd}(A, A^+)$ in place of $\mathrm{Spa}(A, A^+)^\lozenge$, and we abbreviate it as $\mathrm{Spd}(A)$ when A^+ is equal to the subring A° of power bounded elements. In particular, $\mathrm{Spd}(\mathbb{Z}_p)$ parametrizes isomorphism classes of untilts, see [SW20, Definition 10.1.3].

For a formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathbb{Z}_p)$, we write $\mathfrak{X}^{\mathrm{ad}}$ for the pre-adic space associated to $\mathfrak{X}$ as in [SW13, Proposition 2.2.1]. We then write $\mathfrak{X}^\lozenge$ as shorthand for $(\mathfrak{X}^{\mathrm{ad}})^\lozenge$.

For a $\mathbb{Z}_p$-scheme X, we can attach to it two different v-sheaves, following [AGLR22, Section 2.2]. If $X = \mathrm{Spec}(A)$ is affine, we define v-sheaves $X^\diamond$ and $X^\Diamond$ whose points on an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ are

$$X^\diamond(S) = \{(\mathrm{Spa}(R^\sharp, R^{\sharp +}), f : A \to R^{\sharp +})\}/\mathrm{isom.},$$

and respectively

$$X^\Diamond(S) = \{(\mathrm{Spa}(R^\sharp, R^{\sharp +}), f : A \to R^\sharp)\}/\mathrm{isom.},$$

where $\mathrm{Spa}(R^\sharp, R^{\sharp+})$ denotes an untilt of $\mathrm{Spa}(R, R^+)$, and in each case f denotes a $\mathbb{Z}_p$-algebra homomorphism.[3]

Both $(-)^\diamond$ and $(-)^\Diamond$ are compatible with localisations and glue to define functors from the category of schemes over $\mathrm{Spec}(\mathbb{Z}_p)$ to the category of v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$. Following [AGLR22], we refer to these as the "small diamond" and "big diamond" functors, respectively. There is a natural transformation

$$j_X : X^\diamond \to X^\Diamond,$$

which is a monomorphism if X is separated over $\mathbb{Z}_p$, an open immersion if X is separated and of finite type over $\mathbb{Z}_p$, and is an isomorphism if X is proper over $\mathbb{Z}_p$.

The two diamond functors can also be obtained by passing first (suitably) from schemes to their attached adic spaces. Indeed, if X is a $\mathbb{Z}_p$-scheme, then $X^\diamond \cong (\widehat{X})^\Diamond$, where $\widehat{X}$ denotes the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ given by the p-adic completion of X. If X is additionally locally of finite type over $\mathrm{Spec}(\mathbb{Z}_p)$, then we denote by X^{ad} the fiber product

$$X^{\mathrm{ad}} = X \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Z}_p)$$

in the sense of [Hub94, Proposition 3.8], and one can check that $X^\Diamond \cong (X^{\mathrm{ad}})^\Diamond$.

Following [PR24, Definition 2.1.9], for a scheme X which is separated and of finite type over $\mathbb{Z}_p$, we will also consider the v-sheaf $X^{\Diamond/}$, defined by gluing $X^\diamond$ to $X^\Diamond_{\mathbb{Q}_p}$[4] along the open immersion $(X^\diamond)_{\mathbb{Q}_p} \to X^\Diamond_{\mathbb{Q}_p}$, that is,

$$X^{\Diamond/} = X^\diamond \sqcup_{(X^\diamond)_{\mathbb{Q}_p}} X^\Diamond_{\mathbb{Q}_p}.$$

All constructions above extend to schemes over the ring of integers $\mathcal{O}_E$ in a finite extension E of $\mathbb{Q}_p$ or $\breve{\mathbb{Q}}_p$. Below we will use these constructions without comment.

2.1.6. We recall the following construction from [SW20][5].

Definition 2.1.7. A *product of geometric points* is the adic spectrum of a perfectoid Huber pair of the form

$$((\prod_{i\in I} C_i^+)[\varpi^{-1}], \prod_{i \in I} C_i^+),$$

where I is a set, and for each $i \in I$,

[3]Note that in [PR24], the notation $(-)^\blacklozenge$ is used in place of $(-)^\diamond$.

[4]The operations of applying $(-)^\Diamond$ and basechange to $\mathbb{Q}_p$ commute, so this notation is unambiguous.

[5]The terminology first appeared in [Gle20].

- C_i is an algebraically closed perfectoid field of characteristic p, and
- C_i^+ is an open, bounded valuation subring of C_i with pseudouniformizer ϖ_i.

Here we give $\prod_i C_i^+$ the ϖ-adic topology, where $\varpi = (\varpi_i)$.

We introduce the following definition.

Definition 2.1.8. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a map of presheaves of groupoids on Perf.

(1) Given a morphism $T \to S$ in Perf and a 2-commutative diagram of solid arrows

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{F} \\ \downarrow & \nearrow & \downarrow \\ S & \longrightarrow & \mathcal{G}, \end{array} \tag{2.1.2}$$

we say that f has *uniquely existing lifts along* $T \to S$ if the map

$$\lambda_f\colon \mathcal{F}(S) \to \mathcal{F}(T) \times_{\mathcal{G}(T)} \mathcal{G}(S).$$

induced by the diagram (2.1.2) is an equivalence of groupoids.

(2) We say f is *proper** if f has uniquely existing lifts along every morphism of the form

$$\coprod_i s_i := \coprod_i \mathrm{Spa}(C_i, \mathcal{O}_{C_i}) \to S := \mathrm{Spa}((\prod_i C_i^+)[\varpi^{-1}], \prod_i C_i^+)$$

where S is a product of geometric points in characteristic p.

Lemma 2.1.9. *Let $f\colon \mathcal{F} \to \mathcal{G}$ and $g\colon \mathcal{G} \to \mathcal{H}$ be maps between presheaves of groupoids on* Perf. *Assume g is proper*. Then f is proper* if and only if $g \circ f$ is proper*.*

Proof. We note that $\lambda_{g\circ f}$ can be identified with the composition

$$\mathcal{F}(S) \xrightarrow{\lambda_f} \mathcal{F}(\textstyle\coprod_i s_i) \times_{\mathcal{G}(\coprod_i s_i)} \mathcal{G}(S) \xrightarrow{\mathrm{id}\times\lambda_g} \mathcal{F}(\coprod_i s_i) \times_{\mathcal{H}(\coprod_i s_i)} \mathcal{H}(S),$$

and so if λ_g is an equivalence, then λ_f is an equivalence if and only if $\lambda_{g\circ f}$ is an equivalence. ☐

Lemma 2.1.10. *If a map $f\colon \mathcal{F} \to \mathcal{G}$ of v-stacks is proper and representable by diamonds, then f is proper*.*

Proof. This is an immediate consequence of [Zha23, Proposition 2.18]. Note that partial properness is used to produce uniquely existing lifts along $\coprod_i \mathrm{Spa}(C_i, \mathcal{O}_{C_i}) \to \coprod_i \mathrm{Spa}(C_i, C_i^+)$. ☐

Lemma 2.1.11. *Given a Cartesian square of presheaves of groupoids on* Perf

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{g'} & \mathcal{F} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{G}' & \xrightarrow{g} & \mathcal{G}, \end{array}$$

(1) *if f is proper*, then f' is proper*, and*

(2) if f' is proper, and for every product of geometric points S in* Perf *the map $\mathcal{G}'(S) \to \mathcal{G}(S)$ induced by g is essentially surjective, then f is proper*.*

Proof. We note that there is a natural commutative diagram

$$\begin{array}{ccccc} \mathcal{F}'(S) & \xrightarrow{\lambda_{f'}} & \mathcal{F}'(\coprod_i s_i) \times_{\mathcal{G}'(\coprod_i s_i)} \mathcal{G}'(S) & \longrightarrow & \mathcal{G}'(S) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g' \times_g g} & & \downarrow{\scriptstyle g} \\ \mathcal{F}(S) & \xrightarrow{\lambda_f} & \mathcal{F}(\coprod_i s_i) \times_{\mathcal{G}(\coprod_i s_i)} \mathcal{G}(S) & \longrightarrow & \mathcal{G}(S) \end{array}$$

where both squares are Cartesian. It is then clear that λ_f being an equivalence implies $\lambda_{f'}$ being an equivalence. Conversely, if $\mathcal{G}'(S) \to \mathcal{G}(S)$ is essentially surjective, then all vertical maps are, and hence $\lambda_{f'}$ being an equivalence implies that λ_f is an equivalence as well. □

2.2. **Some Bruhat–Tits theory.** Let G be a connected reductive group over $\mathbb{Q}_p$. We write Γ_p for the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and let $I_p \subset \Gamma_p$ be the inertia subgroup. Let $\pi_1(G)$ be the algebraic fundamental group of G, see [Bor98]. Recall from [Kot97, Section 7], that there is a functorial and surjective homomorphism

$$\tilde{\kappa}_G \colon G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_{I_p}.$$

The map $\tilde{\kappa}_G$ is called the Kottwitz map, an exposition of whose construction is given in [KP23, Section 11.5]. Denote the composition of $\tilde{\kappa}_G$ with $\pi_1(G)_{I_p} \to \pi_1(G)_{\Gamma_p}$ by κ_G. We define $G(\breve{\mathbb{Q}}_p)^0$ to be the kernel of $\tilde{\kappa}_G$ and $G(\breve{\mathbb{Q}}_p)^1$ to be the inverse image under $\tilde{\kappa}_G$ of the torsion subgroup $\pi_1(G)_{I_p,\mathrm{tors}}$ of $\pi_1(G)_{I_p}$.

2.2.1. Let $\mathcal{B}(G, \mathbb{Q}_p)$ (resp. $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$) denote the (reduced) Bruhat–Tits building of G (resp. of $G_{\breve{\mathbb{Q}}_p}$); it is a contractible metric space with an action of $G(\mathbb{Q}_p)$ (resp. $G(\breve{\mathbb{Q}}_p)$) by isometries, see [KP23, Axiom 4.1.1, Corollary 4.2.9]. It also naturally has the structure of a polysimplicial complex (see [KP23, Definition 1.5.1]) with facets denoted by $\mathcal{F} \subset \mathcal{B}(G, \mathbb{Q}_p)$ (resp. $\mathcal{F} \subset \mathcal{B}(G, \breve{\mathbb{Q}}_p)$). Note that there is a $G(\mathbb{Q}_p)$-equivariant inclusion $\mathcal{B}(G, \mathbb{Q}_p) \subset \mathcal{B}(G, \breve{\mathbb{Q}}_p)$ identifying $\mathcal{B}(G, \mathbb{Q}_p)$ with the fixed points of $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$ under the Frobenius σ, see [KP23, Theorem 9.2.7].

Given a subset $\Omega \subseteq \mathcal{B}(G, \breve{\mathbb{Q}}_p)$ we consider the pointwise stabilizers $G(\breve{\mathbb{Q}}_p)^0_\Omega$ and $G(\breve{\mathbb{Q}}_p)^1_\Omega$ of Ω inside of $G(\breve{\mathbb{Q}}_p)^0$ and $G(\breve{\mathbb{Q}}_p)^1$, respectively. Subgroups of the form $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}}$ for a facet $\mathcal{F}$ are called *parahoric subgroups.* Following [PR22, Section 2.2], we will define a *quasi-parahoric subgroup* $\breve{K} \subseteq G(\breve{\mathbb{Q}}_p)$ to be any subgroup for which there exists a facet $\mathcal{F}$ such that

$$G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}} = G(\breve{\mathbb{Q}}_p)^0 \cap \mathrm{Stab}_{\mathcal{F}} \subset \breve{K} \subset G(\breve{\mathbb{Q}}_p)^1 \cap \mathrm{Stab}_{\mathcal{F}},$$

where now $\mathrm{Stab}_{\mathcal{F}}$ is the stabilizer of $\mathcal{F}$ in $G(\breve{\mathbb{Q}}_p)$ (rather than the pointwise stabilizer).

2.2.2. For a quasi-parahoric subgroup $\breve{K}$ there is a unique smooth affine group scheme $\mathcal{G}$ over $\breve{\mathbb{Z}}_p$ together with an isomorphism $\mathcal{G}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} G_{\breve{\mathbb{Q}}_p}$ which identifies $\mathcal{G}(\breve{\mathbb{Z}}_p)$ with $\breve{K}$, called the *quasi-parahoric group scheme* associated to $\breve{K}$. When $\breve{K}$ is moreover stable under σ, the group $\mathcal{G}$ descends canonically to a smooth affine group scheme over $\mathbb{Z}_p$. For example, if $\mathcal{F}$ is a facet of $\mathcal{B}(G, \mathbb{Q}_p)$, then the stabilizers $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}}$ and $G(\breve{\mathbb{Q}}_p)^1_{\mathcal{F}}$ are stable under σ, and define quasi-parahoric group schemes over $\mathbb{Z}_p$.

2.2.3. For any quasi-parahoric subgroup $\breve{K} \subseteq G(\breve{\mathbb{Q}}_p)$, there exists by definition a facet $\mathcal{F}$ in $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$ with $G(\breve{\mathbb{Q}}_p)^0 \cap \mathrm{Stab}_{\mathcal{F}} \subseteq \breve{K} \subseteq G(\breve{\mathbb{Q}}_p)^1 \cap \mathrm{Stab}_{\mathcal{F}}$. Intersecting with $G(\breve{\mathbb{Q}}_p)^0$, we observe that $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}} = \breve{K} \cap G(\breve{\mathbb{Q}}_p)^0$, and hence the facet $\mathcal{F}$ is in fact uniquely determined by $\breve{K}$, since the facet is determined by $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}}$, see [KP23, Proposition 9.3.25]. The inclusion $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}} \hookrightarrow \breve{K}$ induces an open immersion $\mathcal{G}^\circ \to \mathcal{G}$ of smooth affine group schemes over $\breve{\mathbb{Z}}_p$, where $\mathcal{G}^\circ$ is the parahoric group scheme corresponding to $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}}$. Moreover, the induced map on the special fiber $\mathcal{G}^\circ \overline{\mathbb{F}}_p \to \mathcal{G}_{\overline{\mathbb{F}}_p}$ is the inclusion of the identity component, see [KP23, Theorem 8.3.13]. By [KP23, Corollary 11.6.3], the finite group

$$\pi_0(\mathcal{G}) := \pi_0(\mathcal{G}_{\overline{\mathbb{F}}_p})$$

can be identified with the image of $\breve{K}$ in $\pi_1(G)_{I_p,\mathrm{tors}}$ under the Kottwitz map $\tilde{\kappa}_G$. When $\breve{K}$ is also stable under σ, we obtain an inclusion $\mathcal{G}^\circ \to \mathcal{G}$ of smooth affine group schemes over $\mathbb{Z}_p$. In this case, $\pi_0(\mathcal{G})$ is a finite étale group scheme over $\mathbb{F}_p$.

Definition 2.2.4. We say that a quasi-parahoric group scheme $\mathcal{G}/\mathbb{Z}_p$ is a *stabilizer Bruhat–Tits group scheme* when $\mathcal{G}(\breve{\mathbb{Z}}_p) \subseteq G(\breve{\mathbb{Q}}_p)$ is of the form $G(\breve{\mathbb{Q}}_p)^1_x$ for a point $x \in \mathcal{B}(G, \mathbb{Q}_p)$ (as opposed to $x \in \mathcal{B}(G, \breve{\mathbb{Q}}_p)$). A *stabilizer parahoric group scheme* is stabilizer Bruhat–Tits group scheme with connected special fiber.

Remark 2.2.5. A subgroup of $G(\breve{\mathbb{Q}}_p)$ being a stabilizer of a point in $\mathcal{B}(G, \mathbb{Q}_p)$ is strictly stronger than being both σ-stable and a stabilizer of a point in $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$.

Following [PR23, Remark 2.3], we consider the group $G = D^\times/\mathbb{G}_m$, where $D/\mathbb{Q}_p$ is the unique quaternion algebra. The building $\mathcal{B}(G, \breve{\mathbb{Q}}_p) \cong \mathcal{B}(\mathrm{PGL}_2, \breve{\mathbb{Q}}_p)$ is a tree, inside which $\mathcal{B}(G, \mathbb{Q}_p)$ is a midpoint of a edge $\mathcal{F}$, see [KP23, Example 9.2.9]. Taking any point $x \in \mathcal{F} \setminus \mathcal{B}(G, \mathbb{Q}_p)$, we see that $G(\breve{\mathbb{Q}}_p)^1_x = G(\breve{\mathbb{Q}}_p)^1_{\mathcal{F}}$ is a σ-stable but not a stabilizer a point in $\mathcal{B}(G, \mathbb{Q}_p)$.

The following result is well known, but we include the proof for the sake of completeness.

Lemma 2.2.6. *Let $\breve{K} \subseteq G(\breve{\mathbb{Q}}_p)$ be a σ-stable quasi-parahoric subgroup. Then there exists a point $x \in \mathcal{B}(G, \mathbb{Q}_p)$ for which $\breve{K} \subseteq G(\breve{\mathbb{Q}}_p)^1_x$ and $\breve{K} \cap G(\breve{\mathbb{Q}}_p)^0 = G(\breve{\mathbb{Q}}_p)^0_x$.*

Proof. Consider the unique facet $\mathcal{F}$ of $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$ for which $G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}} \subseteq \breve{K} \subseteq G(\breve{\mathbb{Q}}_p)^1 \cap \mathrm{Stab}_{\mathcal{F}}$. Since $\breve{K}$ is σ-stable, so is $\mathcal{F}$. We now note that $\mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p) \rtimes \breve{K}$ acts on $\mathcal{F}$ through affine-linear automorphisms. Thus if we take x to be the center-of-mass of the vertices of $\mathcal{F}$, the point x is fixed under the action of $\mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p) \rtimes \breve{K}$. It is also contained in $\mathcal{F}$ because $\mathcal{F}$ is the interior of a convex polytope.

Because x is fixed under $\mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p)$, we have $x \in \mathcal{B}(G, \mathbb{Q}_p)$. Because x is fixed under the action of $\breve{K}$, we have $\breve{K} \subseteq G(\breve{\mathbb{Q}}_p)^1_x$. Finally, we have

$$\breve{K} \cap G(\breve{\mathbb{Q}}_p)^0 = G(\breve{\mathbb{Q}}_p)^0_{\mathcal{F}} = G(\breve{\mathbb{Q}}_p)^0_x$$

because $x \in \mathcal{F}$, see [KP23, Axiom 4.1.20(1)]. □

Corollary 2.2.7. *Let $\mathcal{G}/\mathbb{Z}_p$ be a quasi-parahoric group scheme for $G/\mathbb{Q}_p$. Then there exists a stabilizer Bruhat–Tits model $\mathcal{H}/\mathbb{Z}_p$ of G such that the identity map on G extends to an open embedding $\mathcal{G} \hookrightarrow \mathcal{H}$.*

Proof. We take $\breve{K} = \mathcal{G}(\breve{\mathbb{Z}}_p)$ in Lemma 2.2.6 and let $\mathcal{H}$ be the quasi-parahoric group corresponding to $G(\breve{\mathbb{Q}}_p)^1_x$. The first condition implies that there exists a map $\mathcal{G} \to \mathcal{H}$, and the second condition implies that it is an open embedding. □

2.2.8. Fix a maximal split torus $S \subset G_{\breve{\mathbb{Q}}_p}$ with centralizer T and normalizer N. By definition, the Iwahori–Weyl group $\widetilde{W}$ associated with S sits in a short exact sequence

$$1 \to T(\breve{\mathbb{Q}}_p)^0 \to N(\breve{\mathbb{Q}}_p) \to \widetilde{W} \to 1,$$

see [HR08, Definition 7]. Let G_{sc} denote the simply connected cover of the derived group G_{der} of G, and let $\widetilde{W}_{\mathrm{sc}}$ be the Iwahori–Weyl group of G_{sc}. There is a short exact sequence

$$1 \to \widetilde{W}_{\mathrm{sc}} \to \widetilde{W} \to \pi_1(G)_{I_p} \to 0.$$

Any choice of an alcove[6] $\mathfrak{a}$ of $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$ in the apartment associated to S determines a splitting of this short exact sequence.

For a $G(\overline{\mathbb{Q}}_p)$-conjugacy class μ of cocharacters, we will use the notation $\mathrm{Adm}(\mu^{-1}) \subset \widetilde{W}$ for the μ^{-1}-admissible subset, defined as in [Rap05, Equation (3.4)].

2.2.9. Assume $\mathcal{G}$ is a quasi-parahoric group scheme over $\mathbb{Z}_p$ determined by a σ-stable quasi-parahoric subgroup $\breve{K} \subset G(\breve{\mathbb{Q}}_p)^1_{\mathcal{F}}$, such that $\mathcal{F}$ is σ-stable. As in [PR22, Section 3], we let $\Pi_{\mathcal{G}}$ be the kernel of $H^1(\mathbb{Z}_p, \mathcal{G}) \to H^1(\mathbb{Q}_p, G)$. By Lemma 3.1.1 of *loc. cit.*, we may identify

$$\Pi_{\mathcal{G}} \cong \ker(\pi_0(\mathcal{G})_\phi \to \pi_1(G)_{\Gamma_p}),$$

where $\phi \in \Gamma_p/I_p$ is the Frobenius.[7] Using this and the exact sequence

$$0 \to \pi_1(G)^\phi_{I_p} \to \pi_1(G)_{I_p} \xrightarrow{1-\phi} \pi_1(G)_{I_p} \to \pi_1(G)_{\Gamma_p} \to 0,$$

we may lift any $\delta \in \Pi_{\mathcal{G}}$ to an element $\dot{\delta} \in \pi_0(\mathcal{G})$, such that $\dot{\delta} = (1-\phi)\gamma$ for some $\gamma \in \pi_1(G)_{I_p}$. Choose a splitting of $\widetilde{W} \to \pi_1(G)_{I_p}$ corresponding to a σ-stable alcove

[6]An alcove is a facet which is maximal for the inclusion relation between facets.

[7]From now on we will sometimes use ϕ instead of σ for the Frobenius, to better match the conventions of [PR22].

$\mathfrak{a}$ of $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$ with $\mathcal{F} \subset \mathfrak{a}$ as in Section 2.2.8 above. By [PR22, Lemma 4.3.1] there is a lift $\dot{\gamma}$ of γ to $N(\breve{\mathbb{Q}}_p)$ such that $\dot{\delta} = \phi(\dot{\gamma})^{-1}\dot{\gamma} \in \mathcal{G}(\breve{\mathbb{Z}}_p)$. We then obtain a quasi-parahoric integral model $\mathcal{G}_\delta$ of G such that

$$\mathcal{G}_\delta(\breve{\mathbb{Z}}_p) = \dot{\gamma}\mathcal{G}(\breve{\mathbb{Z}}_p)\dot{\gamma}^{-1} \quad \text{and} \quad \mathcal{G}_\delta^\circ(\breve{\mathbb{Z}}_p) = \dot{\gamma}\mathcal{G}^\circ(\breve{\mathbb{Z}}_p)\dot{\gamma}^{-1}.$$

The $G(\mathbb{Q}_p)$-conjugacy class of $\mathcal{G}_\delta(\breve{\mathbb{Z}}_p)$ does not depend on the choice of $\dot{\gamma}$ or γ, see [PR22, Proposition 4.3.2], and hence the integral model $\mathcal{G}_\delta$ only depends on $\delta \in \Pi_{\mathcal{G}}$ up to isomorphism. However, we shall fix a choice of $\dot{\gamma}$ for each $\delta \in \Pi_{\mathcal{G}}$, for later use in Section 3.3.

2.3. **Finite étale covers of v-sheaves.** Let Λ be a finite abelian group. In this section we will compare Λ-torsors over the v-sheaves associated to schemes with Λ-torsors over the corresponding schemes.

Proposition 2.3.1. *Let X be a flat normal scheme which is separated and of finite type over $\mathbb{Z}_p$, and let $f : Z_{\mathrm{rat}} \to X_{\mathbb{Q}_p}$ be a Λ-torsor. Suppose $\mathscr{X} \to X^{\diamondsuit/}$ is an étale Λ-torsor whose generic fiber is $Z_{\mathrm{rat}}^\diamondsuit \to X_{\mathbb{Q}_p}^\diamondsuit$. Then the relative normalization Z of X in $Z_{\mathrm{rat}} \to X_{\mathbb{Q}_p} \to X$ is an étale Λ-torsor, and $Z^{\diamondsuit/}$ is isomorphic to $\mathscr{X}$ over $X^{\diamondsuit/}$.*

Remark 2.3.2. For X as in the statement of Proposition 2.3.1, we expect that any finite étale cover of $X_{\mathbb{Q}_p}$ which extends to a finite étale cover of $X^{\diamondsuit/}$ comes from a finite étale cover of X. To prove this, it would suffice to prove an analogue of Lemma 2.3.3 below for arbitrary finite étale covers. One would like to apply [Gle20, Theorem 4.27] here, but we were unable to verify that if $\mathcal{F} \to X^{\diamondsuit/}$ is a finite étale cover, that then $\mathcal{F}$ must be a prekimberlite (in the sense of [Gle20, Definition 4.15]). To be precise, we were unable to prove that $\mathcal{F}$ is v-specializing in the sense of [Gle20, Definition 4.6].

We first introduce some notation: Consider the closed and open subschemes of X given by its special and generic fiber

$$X_0 \xrightarrow{i} X \xleftarrow{j} X_{\mathbb{Q}_p}.$$

Let $\widehat{X}$ be the completion of X along X_0, and $\widehat{X}_\eta$ be its adic generic fiber. Its attached diamond $\widehat{X}_\eta^\diamondsuit$ is a quasicompact open sub-diamond of $X_{\mathbb{Q}_p}^\diamondsuit$. Similarly, we denote by $\widehat{Z}$ the p-adic completion of Z.

Lemma 2.3.3. *The natural map*

$$H^1_{\text{ét}}(\widehat{X}, \Lambda) \to H^1_{\text{ét}}(\widehat{X}^\diamondsuit, \Lambda),$$

is an isomorphism.

Proof. We have a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(X_0, \Lambda) & \xleftarrow[\sim]{i^*} & H^1_{\text{ét}}(\widehat{X}, \Lambda) \\ \downarrow\wr & & \downarrow \\ H^1_{\text{ét}}(X_0^\diamond, \Lambda) & \xleftarrow{i^{\diamond,*}} & H^1_{\text{ét}}(\widehat{X}^\diamondsuit, \Lambda). \end{array}$$

The left vertical arrow of the diagram is an isomorphism by [Kim24, Theorem 1.3], and the top arrow is an isomorphism by [Sta24, Tag 0DEG] and [Sta24, Tag 0DEA]. This implies surjectivity of the bottom map $i^{\diamond,*}$. To prove the lemma, it suffices to prove that the bottom arrow is also injective.

Without loss of generality, we may assume that $\widehat{X}$ is connected. This implies that $\widehat{X}^\lozenge$ is connected; indeed, this follows because the map

$$\mathrm{Hom}(\widehat{X}, \mathrm{Spf}\,\mathbb{Z}_p \coprod \mathrm{Spf}\,\mathbb{Z}_p) \to \mathrm{Hom}(\widehat{X}^\lozenge, \mathrm{Spd}\,\mathbb{Z}_p \coprod \mathrm{Spd}\,\mathbb{Z}_p)$$

is a bijection by full-faithfulness of $\widehat{X} \mapsto \widehat{X}^\lozenge$, see [AGLR22, Theorem 2.16]. Suppose an étale Λ-torsor $f : \mathscr{Y} \to \widehat{X}^\lozenge$ splits over $X_0^\diamond$. The natural map from a connected component $\mathscr{G}$ of $\mathscr{Y}$ to $\widehat{X}^\lozenge$ is still finite étale. Thus its image on topological spaces is open and closed, and since $|\widehat{X}^\lozenge|$ is connected, we find that $|\mathscr{G}| \to |X^\diamond|$ is surjective. Since $\mathscr{G} \to X^\diamond$ is quasicompact, it follows from [Sch17, Lemma 12.11] that it is surjective as a map of v-sheaves. Thus the natural map $\mathscr{G} \to \widehat{X}^\lozenge$ is a finite étale cover of $\widehat{X}^\lozenge$.

For $\ell \neq p$, we have

$$\begin{aligned} H^0_{\text{ét}}(\mathscr{Y}, \mathbb{F}_\ell) &\cong H^0_{\text{ét}}(\widehat{X}^\lozenge, f_*\mathbb{F}_\ell) \\ &\cong H^0_{\text{ét}}(X_0^\diamond, i^{\diamond,*} f_*\mathbb{F}_\ell) \\ &\cong H^0_{\text{ét}}(\mathscr{Y}^{\mathrm{red},\diamond}, \mathbb{F}_\ell) \\ &\cong \mathbb{F}_\ell^{\oplus|\Lambda|}. \end{aligned}$$

Here the second isomorphism follows from [FS21, Remark V.4.3(ii)] and the third isomorphism follows from proper base change [Sch17, Theorem 19.2]. Hence $\mathscr{Y}$ has $n := |\Lambda|$ connected components and the map $|\mathscr{Y}| \to |\widehat{X}^\lozenge|$ has fibers of size n. It follows that $|\mathscr{G}| \to |\widehat{X}^\lozenge|$ has fibers of size 1 (since for each connected component the map on topological spaces is surjective), and thus $\mathscr{G} \to \widehat{X}^\lozenge$ is an isomorphism by [Sch17, Lemma 12.5]. It is now clear that the action map $\widehat{X}^\lozenge \times \Lambda \to \mathscr{Y}$ over $\widehat{X}^\lozenge$ is an isomorphism; the injectivity of $i^{\diamond,*}$ follows. □

Proof of Proposition 2.3.1. It follows from [Kim24, Theorem 1.3], as explained in Lemma 2.3.3, that there exists an étale Λ-torsor $\mathfrak{Z} \to \widehat{X}$ whose special fiber identifies with $Z_0 \to X_0$. It thus suffices to show that $\mathfrak{Z} \to \widehat{X}$ is isomorphic to $\widehat{Z}$ over $\widehat{X}$.

By [Sta24, Tag 035L], the relative normalization Z of X in $Z_{\mathrm{rat}} \to X$ is normal, since Z_{rat} is normal. We first prove that $\mathfrak{Z}$ and $\widehat{Z}$ are both η-normal in the sense of [ALY22, Definition A.1]. To show this for $\widehat{Z}$, we use [ALY22, Lemma A.2], which implies that it is enough to check that the local rings of $\widehat{Z}$ at closed points are normal. Note that the closed points of $\widehat{Z}$ are the same as those for Z, and at such a point z we have an isomorphism $\widehat{\mathcal{O}}_{\widehat{Z},z} \xrightarrow{\sim} \widehat{\mathcal{O}}_{Z,z}$. Normality of $\widehat{\mathcal{O}}_{\widehat{Z},z}$ then follows from normality of $\widehat{\mathcal{O}}_{Z,z}$ which in turn follows from the normality of $\mathcal{O}_{Z,z}$. Indeed, the normality of (quasi-excellent) Noetherian local rings is preserved under completion, see [Sta24, Tag 0C23]. It then follows from faithful flatness of $\mathcal{O}_{\widehat{Z},z} \to \widehat{\mathcal{O}}_{\widehat{Z},z}$ along with [Sta24, Tag 033G] that $\mathcal{O}_{\widehat{Z},z}$ is normal. The same proof shows that $\widehat{X}$ is η-normal, and then it follows from [ALY22, Corollary A.16] that the same holds for $\mathfrak{Z}$.

By [ALY22, Lemma 4.1], since both $\mathfrak{Z}$ and $\widehat{Z}$ are η-normal, to prove $\mathfrak{Z}$ is isomorphic to $\widehat{Z}$ over $\widehat{X}$, it suffices to show their rigid generic fibers are isomorphic as étale Λ-torsors over $\widehat{X}_\eta$. In turn, it suffices to show the two Λ-torsors are represented by the same class in $H^1_{\text{ét}}(\widehat{X}_\eta, \Lambda) \cong H^1_{\text{ét}}(\widehat{X}^\lozenge_\eta, \Lambda)$ (see [Sch17, Lemma 15.6] for this isomorphism). But this follows from the commutative diagram below

$$\begin{array}{ccc} H^1_{\text{ét}}(\widehat{X}, \Lambda) & \longrightarrow & H^1_{\text{ét}}(\widehat{X}_\eta, \Lambda) \\ \downarrow\wr & & \downarrow\wr \\ H^1_{\text{ét}}(\widehat{X}^\lozenge, \Lambda) & \longrightarrow & H^1_{\text{ét}}(\widehat{X}^\lozenge_\eta, \Lambda). \end{array}$$

Indeed, going clockwise from $H^1_{\text{ét}}(\widehat{X}, \Lambda)$ to $H^1_{\text{ét}}(\widehat{X}^\lozenge_\eta, \Lambda)$ the class of $\mathfrak{Z} \to \widehat{X}$ is sent to that of $\mathfrak{Z}^\lozenge_\eta \to \widehat{X}^\lozenge_\eta$. On the other hand, by the proof of Lemma 2.3.3, going counterclockwise we get the class of $\mathscr{Z} \times_{X^{\lozenge/}} \widehat{X}^\lozenge_\eta \to \widehat{X}^\lozenge_\eta$. Hence we are done if we can show $\mathscr{Z} \times_{X^{\lozenge/}} \widehat{X}^\lozenge_\eta$ is isomorphic to $\widehat{Z}^\lozenge_\eta$. But since $Z \to X$ is integral, this follows from [Hub96, Proposition 1.9.6] and our assumption that the generic fiber of $\mathscr{Z} \to X^{\lozenge/}$ is given by $Z^\lozenge_{\text{rat}} \to X^\lozenge_{\mathbb{Q}_p}$. □

3. The moduli stack of quasi-parahoric shtukas

The goal of this section is to study moduli stacks of quasi-parahoric shtukas, and to prove Corollary 3.3.7.

3.1. **Newton strata in the moduli stack of quasi-parahoric shtukas.** In what follows we let $\mathcal{G}$ be a quasi-parahoric group scheme over $\mathbb{Z}_p$ with generic fiber G, and we let $\mathcal{G}^\circ \subset \mathcal{G}$ be the corresponding parahoric group scheme. Let $\mathrm{Gr}_{\mathcal{G}}$ and $\mathrm{Gr}_{\mathcal{G}^\circ}$ over $\mathrm{Spd}(\mathbb{Z}_p)$ be the Beilinson–Drinfeld affine Grassmannians of [SW20, Definition 20.3.1]. The natural map $\mathrm{Gr}_{\mathcal{G}^\circ} \to \mathrm{Gr}_{\mathcal{G}}$ becomes an isomorphism after base changing to $\mathrm{Spd}(\mathbb{Q}_p)$. We will call this common base change the B^+_{dR}-affine Grassmannian, and we will denote it by $\mathrm{Gr}_G \to \mathrm{Spd}(\mathbb{Q}_p)$, see [SW20, Lecture XIX].

3.1.1. For a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of minuscule cocharacters μ of G with reflex field E, we denote by $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,E}$ the closed Schubert-cell determined by μ, see [SW20, Section 19.2]. We define the *v-sheaf local model* $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}(\mathcal{O}_E)}$ to be the v-sheaf theoretic closure of $\mathrm{Gr}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}(\mathcal{O}_E)}$, and similarly we have $\mathbb{M}^{\mathrm{v}}_{\mathcal{G}^\circ,\mu}$. As shown in [SW20, Proposition 21.4.3], functoriality of local models applied to the map $\mathcal{G}^\circ \to \mathcal{G}$ induces an isomorphism

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G}^\circ,\mu} \xrightarrow{\sim} \mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}. \tag{3.1.1}$$

3.1.2. A $\mathcal{G}$-shtuka over a perfectoid space S with leg at an untilt $S^\sharp$ is defined to be a $\mathcal{G}$-torsor $\mathscr{P}$ over $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$, together with an isomorphism of $\mathcal{G}$-torsors[8]

$$\phi_{\mathscr{P}} : \mathrm{Frob}_S^* \mathscr{P}\big|_{S\dot{\times}\mathrm{Spa}(\mathbb{Z}_p)\setminus S^\sharp} \to \mathscr{P}\big|_{S\dot{\times}\mathrm{Spa}(\mathbb{Z}_p)\setminus S^\sharp},$$

that is meromorphic along $S^\sharp$ in the sense of [SW20, Definition 5.3.5]. We will occasionally denote such a meromorphic map by

$$\phi_{\mathscr{P}} : \mathrm{Frob}_S^* \mathscr{P} \dashrightarrow \mathscr{P},$$

when the choice of untilt $S^\sharp$ is clear. For μ as above, we say that a $\mathcal{G}$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ is bounded by μ if the relative position of $\mathrm{Frob}_S^* \mathscr{P}$ and $\mathscr{P}$ at $S^\sharp$ is bounded by the v-sheaf local model $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}(\mathcal{O}_E)}$, see [PR24, Section 2.3.4].

For S in Perf, denote by $\mathrm{Sht}_{\mathcal{G}}(S)$ the groupoid of triples $(S^\sharp, \mathscr{P}, \phi_{\mathscr{P}})$, where $S^\sharp$ is an untilt of S and where $(\mathscr{P}, \phi_{\mathscr{P}})$ is a $\mathcal{G}$-shtuka over S with leg at $S^\sharp$. By [SW20, Proposition 2.1.2], the assignment $S \mapsto \mathrm{Sht}_{\mathcal{G}}(S)$ defines a v-stack $\mathrm{Sht}_{\mathcal{G}}$ on Perf (for this, use the fact that $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ is sousperfectoid by the proof of [SW20, Proposition 11.2.1]). For μ as above, we let $\mathrm{Sht}_{\mathcal{G},\mu} \subset \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathcal{O}_E)$ be the closed substack whose S-points consists of $\mathcal{G}$-shtukas over S with one leg at $S^\sharp$, which are bounded by μ.[9][10]

3.1.3. Let $S = \mathrm{Spa}(R, R^+) \to \mathrm{Spd}(\mathbb{Z}_p)$ be an object in Perf together with an untilt $S^\sharp$, and let $(\mathscr{P}, \phi_{\mathscr{P}})$ be a $\mathcal{G}$-shtuka over S with one leg at $S^\sharp$. We can choose r sufficiently large such that $\mathcal{Y}_{[r,\infty)}$ does not meet the divisor of $\mathcal{Y}_{[r,\infty)}$ defined by $S^\sharp$. The restriction of $(\mathscr{P}, \phi_{\mathscr{P}})$ determines a $\phi = \mathrm{Frob}_S$-equivariant $\mathcal{G}$-torsor on $\mathcal{Y}_{[r,\infty)}$. By spreading out via the Frobenius (see [SW20, Proposition 22.1.1]), the bundle $\mathscr{P}$ descends to a G-bundle $\mathcal{E}(\mathscr{P}, \phi_{\mathscr{P}})$ on X_S. In this way we obtain a morphism of v-stacks on Perf

$$\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}_G, \quad (\mathscr{P}, \phi_{\mathscr{P}}) \mapsto \mathcal{E}(\mathscr{P}, \phi_{\mathscr{P}}).$$

and we will denote both this map and its restriction to $\mathrm{Sht}_{\mathcal{G},\mu}$ by BL°.

Using BL°, we obtain locally closed substacks $\mathrm{Sht}_{\mathcal{G}}^{[b]} \subset \mathrm{Sht}_{\mathcal{G}}$ and $\mathrm{Sht}_{\mathcal{G},\mu}^{[b]} \subset \mathrm{Sht}_{\mathcal{G},\mu}$ defined as in (2.1.1). We will refer to these as the Newton strata corresponding to $[b]$ in $\mathrm{Sht}_{\mathcal{G}}$ and $\mathrm{Sht}_{\mathcal{G},\mu}$, respectively.

[8] Here we consider $\mathrm{Frob}_S^*(\mathcal{G})$ as a $\mathcal{G}$-torsor via the isomorphism $\mathrm{Frob}_S^*(\mathcal{G}) \to \mathcal{G}$ coming from the fact that $\mathcal{G}$ is defined over $\mathbb{Z}_p$.

[9] Our moduli stacks $\mathrm{Sht}_{\mathcal{G},\mu}$ should not be confused with the moduli spaces of shtukas $\mathrm{Sht}_{(\mathcal{G},b,\mu)}$ of [SW20, Definition 23.1.1], also denoted by $\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))}$ in [GLX23, Section 3.4]. They should also not be confused with the moduli spaces of p-adic shtukas $\mathrm{Sht}_{\mathbb{Z}_p}^{\mathcal{G}_b}$ of [Gle21, Definition 2.21]. These objects are moduli spaces of shtukas with a framing (towards b); the similarity in notation is unfortunate.

[10] The stack denoted by $\mathrm{Sht}_{\mathcal{G}}$ in [GI23, Definition 7.4 of version 1] corresponds in our notation to the stack $\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{F}_p)$.

3.1.4. For ℓ an algebraically closed field in characteristic p together with a fixed embedding $e\colon k_E \hookrightarrow \ell$, write

$$W_{\mathcal{O}_E,e}(\ell) = \mathcal{O}_E \otimes_{W(k_E),e} W(\ell).$$

We make the following definition, which is a slight generalization of [SW20, Definition 25.1.1].

Definition 3.1.5. Let ℓ be a perfect field of characteristic p together with an embedding $e\colon k_E \hookrightarrow \ell$, and let $b \in G(W(\ell)[p^{-1}])$. The *integral local Shimura variety*

$$\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,e} \to \operatorname{Spd} W_{\mathcal{O}_E,e}(\ell)$$

is the v-sheaf on Perf that assigns to each perfectoid S the set of isomorphism classes of tuples $(S^\sharp, \mathscr{P}, \phi_{\mathscr{P}}, \iota_r)$, where $S^\sharp$ is an untilt of S over $W_{\mathcal{O}_E,e}(\ell)$, where $(\mathscr{P}, \phi_{\mathscr{P}})$ is a $\mathcal{G}$-shtuka with one leg along $S^\sharp$ bounded by μ, and ι_r is an isomorphism

$$\iota_r\colon G|_{\mathcal{Y}_{[r,\infty)}(S)} \xrightarrow{\cong} \mathscr{P}|_{\mathcal{Y}_{[r,\infty)}(S)}$$

for $r \gg 0$, which satisfies $\iota_r \circ \phi_{\mathscr{P}} = (b \times \mathrm{Frob}_S) \circ \iota_r$. Two tuples $(S^\sharp, \mathscr{P}, \phi_{\mathscr{P}}, \iota_r)$, $(S^\sharp, \mathscr{P}', \phi'_{\mathscr{P}}, \iota'_{r'})$ are isomorphic if there is an isomorphism of $\mathcal{G}$-shtukas $(\mathscr{P}, \phi_{\mathscr{P}}) \to (\mathscr{P}', \phi'_{\mathscr{P}})$ which is compatible with ι_r and $\iota'_{r'}$ after restricting to $\mathcal{Y}_{[R,\infty)}(S)$ for some $R \gg r, r'$.

Lemma 3.1.6. *There is a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,e} & \longrightarrow & \mathrm{Sht}^{[b]}_{\mathcal{G},\mu} \times_{\mathcal{O}_E} W_{\mathcal{O}_E,e}(\ell) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{BL}^\circ} \\ \operatorname{Spd}(\ell) & \xrightarrow{\quad b \quad} & \mathrm{Bun}^{[b]}_G, \end{array}$$

where $b : \operatorname{Spd}(\ell) \to \mathrm{Bun}^{[b]}_G$ *is the map coming from* [Ans23, Theorem 5.3].

Proof. This follows from unwinding the definition of the map $\mathrm{BL}^\circ : \mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Bun}_G$ and the definition of the sheaf $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,e}$. ☐

Remark 3.1.7. If

$$b \in \bigcup_{w \in \mathrm{Adm}(\mu^{-1})} \mathcal{G}(W(\ell)) w \mathcal{G}(W(\ell)),$$

then b defines an element $b \in \mathrm{Sht}_{\mathcal{G},\mu}(\operatorname{Spd}(\ell))$ lifting $b \in \mathrm{Bun}_G(\operatorname{Spd}(\ell))$, see [PR24, Remark 4.2.3]. The universal property of the fiber product diagram of Lemma 3.1.6 then gives us a tautological base point $x_0 : \operatorname{Spd}(\ell) \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,e}$.

3.1.8. We observe that it follows from the argument in [Zha23, Proposition 11.16][11] that there is an isomorphism (here the sheaf $\underline{\mathcal{G}^\circ(\mathbb{Z}_p)}$ is as in [Sch17, the discussion

[11]Note that although [Zha23, Proposition 11.16] assumes $\mathcal{G}$ being reductive, the reference to [SW20] cited in *loc. cit.* only assumes that $\mathcal{G}$ is smooth and has connected special fiber, so the argument works verbatim.

before Definition 10.12])

$$c : \mathrm{Sht}_{\mathcal{G}^\circ,\mu,E} \to \left[\mathrm{Gr}_{G,\mu^{-1}}/\underline{\mathcal{G}^\circ(\mathbb{Z}_p)}\right]. \tag{3.1.2}$$

We generally do *not* have an isomorphism as in (3.1.2) for $\mathcal{G}$-shtukas, as we will explain below in Corollary 3.3.9.

Lemma 3.1.9. *The map*

$$\mathrm{BL}^\circ : \mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Bun}_G$$

factors through $\mathrm{Bun}_{G,\mu^{-1}}$

Proof. By definition, $\mathrm{Bun}_{G,\mu^{-1}}$ is the subfunctor of Bun_G consisting of maps $X \to \mathrm{Bun}_G$ for which $|X| \to |\mathrm{Bun}_G| \cong B(G)$ factors over $B(G,\mu^{-1})$, see Section 2.1.2. It is therefore enough to show the factorization at the level of topological spaces. By Lemma 3.1.6 and the v-surjectivity of $b : \mathrm{Spd}(\overline{\mathbb{F}}_p) \to \mathrm{Bun}_G^{[b]}$, it suffices to show that $|\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b,\mu,e}|$ is empty unless $[b] \in B(G,\mu^{-1})$.

By [PR24, Theorem 3.3.3], see [Gle21], the reduction $\left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b,\mu,e}\right)^{\mathrm{red}}$ in the sense of [Gle20, Definition 3.12] is isomorphic to the affine Deligne–Lusztig variety $X_{\mathcal{G}^\circ}(b,\mu^{-1})$ (see [PR24, Definition 3.3.1]). The space $X_{\mathcal{G}^\circ}(b,\mu^{-1})$ is empty unless $[b] \in B(G,\mu^{-1})$ by [He16, Theorem A] and this along with [Gle20, Proposition 4.8.(4)] implies that $|\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b,\mu,e}|$ is empty unless $[b] \in B(G,\mu^{-1})$. □

Lemma 3.1.9 will generally *not* be true for $\mathrm{Sht}_{\mathcal{G},\mu}$, but we do have the following result: We recall from [FS21, Theorem III.2.7] that there is a locally constant map

$$\kappa_G : |\mathrm{Bun}_G| \to \pi_1(G)_{\Gamma_p},$$

such that $\mathrm{Bun}_{G,\mu^{-1}}$ maps to $-\mu^\natural = \kappa_G(\mu^{-1})$. Here $\pi_1(G)$ is the algebraic fundamental group of G and $\Gamma_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We let $\mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu} \subset \mathrm{Sht}_{\mathcal{G},\mu}$ be the open and closed substack that is the preimage of $-\mu^\natural$ under κ_G.

Proposition 3.1.10. *The map* $\mathrm{BL}^\circ : \mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu} \to \mathrm{Bun}_G$ *factors through* $\mathrm{Bun}_{G,\mu^{-1}}$.

In the proof, we use the notation from Section 2.2.9.

Proof. Since $\mathrm{Bun}_{G,\mu^{-1}}$ is an open substack of Bun_G, it is enough to check that for any map $\mathrm{Spa}(C,C^+) \to \mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu}$ with C an algebraically closed perfectoid field, the induced map $|\mathrm{Spa}(C,C^+)| \to B(G)$ on topological spaces has image contained in $B(G,\mu^{-1})$. Using [PR24, Proposition 2.1.1], we see that the restriction map

$$\mathrm{Bun}_G(\mathrm{Spa}(C,C^+)) \to \mathrm{Bun}_G(\mathrm{Spa}(C,\mathcal{O}_C))$$

is an equivalence, and thus we may assume that $C^+ = \mathcal{O}_C$. Recall that $\mathrm{Sht}_{\mathcal{G},\mu}$ has a map to $\mathrm{Spd}(\mathbb{Z}_p)$. We will verify the statement by dividing into the case when $C^\sharp$ has characteristic zero and when $C^\sharp = C$ has characteristic p.

Case 1: First assume that $C^\sharp$ has characteristic zero, and let $(\mathscr{P},\phi_{\mathscr{P}})$ be a $\mathcal{G}$-shtuka with leg at $C^\sharp$ bounded by μ. For sufficiently small r, the restriction $\mathscr{P}|_{\mathcal{Y}_{[0,r]}(C,\mathcal{O}_C)}$

defines a shtuka with no leg, and by [KL15, Theorem 8.5.3], this corresponds to an exact tensor functor $\mathsf{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathsf{Mod}_{\mathbb{Z}_p}$ which gives an element of $H^1_{\mathrm{et}}(\mathbb{Z}_p, \mathcal{G})$ by the Tannakian interpretation of torsors. Under the map (where $B(G)_{\mathrm{bsc}} \subset B(G)$ denotes the subset of basic elements)

$$H^1_{\mathrm{et}}(\mathbb{Z}_p, \mathcal{G}) \to H^1_{\mathrm{et}}(\mathbb{Q}_p, G) \hookrightarrow B(G)_{\mathrm{bsc}},$$

this determines the isomorphism class $[b_0] \in B(G)_{\mathrm{bsc}}$ of the C-point of the G-bundle on $X_{(C,\mathcal{O}_C)}$ coming from $\mathscr{P}|_{\mathcal{Y}_{(0,r]}(C,\mathcal{O}_C)}$, see [SW20, Section 22.3]. The G-bundle $\mathcal{E}(\mathscr{P}, \phi_{\mathscr{P}})$ comes from restricting to $\mathcal{Y}_{[R,\infty)}(C, \mathcal{O}_C)$ for $R \gg 0$, let us denote its isomorphism class by $[b] \in B(G)$. Then $\kappa([b_0]) - \kappa([b]) = -\mu^\natural$ as they are related by a modification bounded by μ, see the proof of [SW20, Proposition 25.3.2]. Since $(\mathscr{P}, \phi_{\mathscr{P}})$ is assumed to be in $\mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu}$, it follows that $\kappa([b_0]) = 0$. Moreover, since $[b_0]$ is basic, we see that $[b_0] = 0$. Therefore $[b] \in B(G, \mu^{-1})$ by [Rap18, Proposition 9].

Case 2: Next, we consider the case where $C^\sharp = C$ has characteristic p. As before, let $(\mathscr{P}, \phi_{\mathscr{P}})$ be a $\mathcal{G}$-shtuka with leg at C bounded by μ. Using [PR24, Proposition 2.1.3] and [PR22, Proposition 3.2.1], we may find a $\mathcal{G}$-torsor $\mathcal{P}$ on $\mathrm{Spec}(W(\mathcal{O}_C))$ together with a meromorphic map $\phi_{\mathcal{P}} \colon \mathcal{P} \to \phi^*\mathcal{P}$ that analytifies to $(\mathscr{P}, \phi_{\mathscr{P}})$. In particular, we may recover $[b] \in B(G)$ also by considering the isomorphism class of the G-isocrystal $(\mathcal{P}_{W(C)[p^{-1}]}, \phi_{\mathcal{P}_{W(C)[p^{-1}]}})$.

Fix a trivialization of $\mathcal{P} \cong \mathcal{G}_{W(\mathcal{O}_C)}$, so that the Frobenius $\phi_{\mathcal{P}_{W(C)[p^{-1}]}}$ gives us an element $b \in G(W(\mathcal{O}_C)[p^{-1}])$. The boundedness by μ condition now tells us that

$$b \in \mathcal{G}^\circ(W(C))\,\mathrm{Adm}(\mu^{-1})\mathcal{G}(W(C)).$$

Indeed, by definition the element b lies in

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}(C, \mathcal{O}_C) \subset \mathrm{Gr}_{\mathcal{G}}(C, \mathcal{O}_C) = G(W(C)[1/p])/\mathcal{G}(W(\mathcal{O}_C)). \tag{3.1.3}$$

By [AGLR22, Theorem 6.16], we may identify

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G}^\circ,\mu}(C, \mathcal{O}_C) \subset G(W(C)[1/p])/\mathcal{G}^\circ(W(\mathcal{O}_C))$$

with

$$\mathcal{G}^\circ(W(C))\,\mathrm{Adm}(\mu^{-1})\mathcal{G}^\circ(W(C)) \subset G(W(C)[1/p])/\mathcal{G}^\circ(W(\mathcal{O}_C)).$$

Since the natural map $\mathrm{Gr}_{\mathcal{G}^\circ} \to \mathrm{Gr}_{\mathcal{G}}$ induces an isomorphism

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G}^\circ,\mu} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu},$$

we may identify (3.1.3) with

$$\mathcal{G}^\circ(W(C))\,\mathrm{Adm}(\mu^{-1})\mathcal{G}(W(C)) \subset G(W(C)[1/p])/\mathcal{G}(W(\mathcal{O}_C)).$$

We fix an embedding $\overline{\mathbb{F}}_p \hookrightarrow C$, so that we have a group homomorphism $G(\breve{\mathbb{Q}}_p) \to G(W(C)[p^{-1}])$. We want to show that the class $[b] \in B(G)$, regarded as a ϕ-conjugacy class in $G(W(C)[p^{-1}])$, is in $B(G, \mu^{-1})$. Recall our assumption that $\kappa_G(b) = -\mu^\natural \in \pi_1(G)_{\Gamma_p}$. This means that

$$\beta = -\tilde{\kappa}_G(b) - [\mu] \in \pi_0(\mathcal{G}) \subseteq \pi_1(G)_{I_p}$$

can be written as $\beta = (1 - \phi)\gamma$ for some $\gamma \in \pi_1(G)_{I_p}$.

Choose a splitting of $\tilde{W} \to \pi_1(G)_{I_p}$ corresponding to a σ-stable alcove $\mathfrak{a}$ of $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$ with $\mathcal{F} \subset \mathfrak{a}$, as in Section 2.2.8.[12] By [PR22, Lemma 4.3.1], there is a lift $\dot{\gamma}$ of γ to $N(\breve{\mathbb{Q}}_p)$ such that $\phi(\dot{\gamma})^{-1}\dot{\gamma} \in \mathcal{G}(\breve{\mathbb{Z}}_p)$. We now have

$$b' := \dot{\gamma} b \phi(\dot{\gamma})^{-1} = \dot{\gamma}(b\phi(\dot{\gamma})^{-1}\dot{\gamma})\dot{\gamma}^{-1} \in \dot{\gamma}\mathcal{G}(W(C)) \operatorname{Adm}(\mu^{-1})\mathcal{G}(W(C))\dot{\gamma}^{-1}.$$

As in the proof of [PR22, Proposition 4.3.4], we may identify

$$\mathcal{G}(W(C)) \operatorname{Adm}(\mu^{-1})\mathcal{G}(W(C))\dot{\gamma}^{-1} = \mathcal{G}_\delta(W(C)) \operatorname{Adm}(\mu^{-1})\mathcal{G}_\delta(W(C)),$$

where $\delta \in \Pi_G$ is the image of β under $\pi_0(\mathcal{G}) \to \pi_0(\mathcal{G})_\phi$, so that $\mathcal{G}_\delta(W(C)) = \dot{\gamma}\mathcal{G}(W(C))\dot{\gamma}^{-1}$. We then have

$$\begin{aligned} \tilde{\kappa}_G(b') &= (1-\phi)\gamma + \tilde{\kappa}_G(b) \\ &= (1-\phi)\gamma - [\mu] - (1-\phi)\gamma \\ &= -[\mu] \in \pi_1(G)_{I_p}. \end{aligned}$$

Moreover, as b' is a ϕ-conjugate of b, it suffices to show that $[b'] \in B(G, \mu^{-1})$. By evaluating the isomorphism $\mathbb{M}^{\mathrm{v}}_{\mathcal{G}_\delta^\circ,\mu} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{G}_\delta,\mu}$ on $(C, \mathcal{O}_C)$-points, we obtain the equality (as above)

$$\mathcal{G}_\delta(W(C)) \operatorname{Adm}(\mu^{-1})\mathcal{G}_\delta(W(C)) = \mathcal{G}_\delta^\circ(W(C)) \operatorname{Adm}(\mu^{-1})\mathcal{G}_\delta(W(C)).$$

Thus we can write $b' = hwg$ with $h \in \mathcal{G}_\delta^\circ(W(C))$, $w \in \operatorname{Adm}(\mu^{-1})$, and $g \in \mathcal{G}_\delta(W(C))$. By applying $\tilde{\kappa}_G$ on both sides, we obtain

$$-[\mu] = \tilde{\kappa}_G(b') = \tilde{\kappa}_G(h) + \tilde{\kappa}_G(w) + \tilde{\kappa}_G(g) = -[\mu] + \tilde{\kappa}_G(g) \in \pi_1(G)_{I_p}$$

This shows that $\tilde{\kappa}_G(g) = 0$, and hence

$$g \in \mathcal{G}_\delta(W(C)) \cap \ker \tilde{\kappa}_G = \mathcal{G}_\delta^\circ(W(C)).$$

It now follows that

$$b' \in \mathcal{G}_\delta^\circ(W(C)) \operatorname{Adm}(\mu^{-1})\mathcal{G}_\delta^\circ(W(C))$$

and thus by [He16, Theorem A] that $[b] = [b'] \in B(G, \mu^{-1})$. □

3.2. **A group action on the moduli stack of parahoric shtukas.** Let $\mathcal{G}/\mathbb{Z}_p$ be a quasi-parahoric group scheme as before. The goal of this section is to construct an action of $\pi_0(\mathcal{G})^\phi$ on $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$ together with a map $[\mathrm{Sht}_{\mathcal{G}^\circ,\mu}/\pi_0(\mathcal{G})^\phi] \to \mathrm{Sht}_{\mathcal{G},\mu}$.

3.2.1. Recall, e.g., from [PR22, Section 4.4], that there is a short exact sequence

$$1 \to \mathcal{G}^\circ(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{Z}_p) \to \pi_0(\mathcal{G})^\phi \to 1.$$

For an element g of $\pi_0(\mathcal{G})^\phi$ and a representation $(\Lambda, \rho\colon \mathcal{G}^\circ \to \mathrm{GL}(\Lambda))$ in $\mathsf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^\circ)$, we define the lattice

$$g\Lambda = \rho(\tilde{g})\Lambda \subseteq \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where $\tilde{g} \in \mathcal{G}(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ is a lift of g. Note that this does not depend on the choice of lift $\tilde{g}$ because Λ is stable under $\mathcal{G}^\circ(\mathbb{Z}_p)$.

[12] Here $\mathcal{F}$ is as in Section 2.2.9.

Lemma 3.2.2. *The group homomorphism* $\rho_{\mathbb{Q}_p}\colon G \to \mathrm{GL}_{\mathbb{Q}_p}(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ *extends (uniquely) to a homomorphism* $g\rho : \mathcal{G}^\circ \to \mathrm{GL}_{\mathbb{Z}_p}(g\Lambda)$.

Proof. Both $\mathcal{G}^\circ$ and $\mathrm{GL}_{\mathbb{Z}_p}(g\Lambda)$ are smooth affine integral models of their respective generic fibers. Therefore by [KP23, Corollary 2.10.10], it suffices to show that the image of $\mathcal{G}^\circ(\breve{\mathbb{Z}}_p)$ is contained in $\mathrm{GL}_{\breve{\mathbb{Z}}_p}(g\Lambda \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p)$. Since $\mathcal{G}^\circ(\breve{\mathbb{Z}}_p) \subseteq \mathcal{G}(\breve{\mathbb{Z}}_p)$ is normal, it is in particular stable under conjugation by $\tilde{g} \in \mathcal{G}(\mathbb{Z}_p)$ a lift of g. The claim now follows, as $\Lambda \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ is stable under the action of $\mathcal{G}^\circ(\breve{\mathbb{Z}}_p)$. □

We see that $\rho \mapsto g\rho$ defines a $\pi_0(\mathcal{G})^\phi$-action on the category $\mathsf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^\circ)$ by exact tensor equivalences. Moreover, for any representation (Λ, ρ) in the image of the forgetful (exact tensor) functor $\mathsf{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathsf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^\circ)$, we have $\Lambda = g\Lambda$ and hence $\rho = g\rho$.

3.2.3. Using the Tannakian formalism, this defines a $\pi_0(\mathcal{G})^\phi$-action on the groupoid of $\mathcal{G}^\circ$-torsors on $\mathcal{Y}_{[0,\infty)}(S)$ for $S \in \mathrm{Perf}$. More precisely, for each $\mathcal{G}^\circ$-torsor $\mathscr{P}$ on $Y_{[0,\infty)}(S)$ and an element $g \in \pi_0(\mathcal{G})^\phi$, there is a $\mathcal{G}^\circ$-torsor $g\mathscr{P}$ so that

$$\mathcal{G}^\circ \backslash ((g\mathscr{P}) \times \Lambda) = \mathcal{G}^\circ \backslash (\mathscr{P} \times g^{-1}\Lambda).$$

By construction, if we push out to $\mathcal{G}$, we see that there is a canonical isomorphism

$$\mathcal{G} \times^{\mathcal{G}^\circ} \mathscr{P} \cong \mathcal{G} \times^{\mathcal{G}^\circ} (g\mathscr{P}).$$

We also see that there is a canonical meromorphic homomorphism

$$\mathscr{P} \overset{g}{\dashrightarrow} g\mathscr{P}$$

with a leg at S, coming from the fact that $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = g^{-1}\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Remark 3.2.4. The $\pi_0(\mathcal{G})^\phi$-action we construct is in fact equivalent to the one given in [PR22, Section 4.4]. If we choose a lift $\tilde{g} \in \mathcal{G}(\mathbb{Z}_p)$ and twist the $\mathcal{G}^\circ(\mathbb{Z}_p)$-torsor structure to get a new torsor $\mathscr{P}_{\tilde{g}}$, then $\tilde{g}^{-1}$ induces an isomorphism

$$\begin{aligned}
\mathcal{G}^\circ \backslash (\mathscr{P}_{\tilde{g}} \times \Lambda) &= (\mathscr{P} \times \Lambda) / \big((\tilde{g}^{-1} h \tilde{g} x, hy) \sim (x, y) \big) \\
&\xrightarrow{(\mathrm{id}, \tilde{g}^{-1})} (\mathscr{P} \times \tilde{g}^{-1}\Lambda) / \big((\tilde{g}^{-1} h \tilde{g} x, \tilde{g}^{-1} hy) \sim (x, \tilde{g}^{-1} y) \big) \\
&= (\mathscr{P} \times \tilde{g}^{-1}\Lambda) / \big((h'x, h'y') \sim (x, y') \big) = \mathcal{G}^\circ \backslash (\mathscr{P} \times \tilde{g}^{-1}\Lambda)
\end{aligned}$$

by substituting $h' = \tilde{g}^{-1} h \tilde{g}$ and $y' = \tilde{g}^{-1} y$.

3.2.5. We now use this action to define an action of $\pi_0(\mathcal{G})^\phi$ on $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$. For $S \in \mathrm{Perf}$ and a $\mathcal{G}^\circ$-torsor $\mathscr{P}$ on $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$, first note that $g(\mathrm{Frob}_S^* \mathscr{P}) = \mathrm{Frob}_S^*(g\mathscr{P})$, as both correspond to the exact tensor functor

$$\mathsf{Rep}(\mathcal{G}^\circ) \xrightarrow{g^{-1}} \mathsf{Rep}(\mathcal{G}^\circ) \to \mathsf{Vect}(S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)) \xrightarrow{\mathrm{Frob}_S^*} \mathsf{Vect}(S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)).$$

Given $(\mathscr{P}, \phi_{\mathscr{P}})$ a $\mathcal{G}^\circ$-shtuka, we now define $\phi_{g\mathscr{P}}$ to be the meromorphic map

$$\mathrm{Frob}_S^*(g\mathscr{P}) = g(\mathrm{Frob}_S^* \mathscr{P}) \overset{g^{-1}}{\dashrightarrow} \mathrm{Frob}_S^* \mathscr{P} \overset{\phi_{\mathscr{P}}}{\dashrightarrow} \mathscr{P} \overset{g}{\dashrightarrow} g\mathscr{P}.$$

Proposition 3.2.6. *Let $S \in \mathrm{Perf}$ and let $(\mathscr{P}, \phi_{\mathscr{P}}) \in \mathrm{Sht}_{\mathcal{G}^\circ,\mu}(S)$ be a $\mathcal{G}^\circ$-shtuka. Then $(g\mathscr{P}, \phi_{g\mathscr{P}})$ defines an object of $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}(S)$.*

Proof. We first check that $\phi_{g\mathscr{P}}$ only has poles at $R^\sharp$. This can be checked by considering the induced map on vector bundle shtukas for each $\Lambda \in \mathsf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^\circ)$. We observe that the map

$$\begin{aligned}\mathcal{G}^\circ\backslash(\mathrm{Frob}_S^*\mathscr{P} \times g^{-1}\Lambda) = \mathcal{G}^\circ\backslash(\mathrm{Frob}_S^*(g\mathscr{P}) \times \Lambda) \dashrightarrow \mathcal{G}^\circ\backslash(\mathrm{Frob}_S^*\mathscr{P} \times \Lambda)\\ \dashrightarrow \mathcal{G}^\circ\backslash(\mathscr{P} \times \Lambda) \dashrightarrow \mathcal{G}^\circ\backslash(g\mathscr{P} \times \Lambda) = \mathcal{G}^\circ\backslash(\mathscr{P} \times g^{-1}\Lambda)\end{aligned}$$

is identified with $\phi_{\mathscr{P}} \times g^{-1}\Lambda$, and therefore only has a pole at $R^\sharp$. Next, we need to show that the modification is bounded by μ. Choosing a lift $\tilde{g} \in \mathcal{G}(\mathbb{Z}_p)$ of g as in Remark 3.2.4, this follows from the stability of $\mathbb{M}^{\mathrm{v}}_{\mathcal{G}^\circ,\mu}$ under conjugation by $\tilde{g}$. □

We define the action of $\pi_0(\mathcal{G})^\phi$ on $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$ by

$$g\colon (\mathscr{P}, \phi_{\mathscr{P}}) \mapsto (g\mathscr{P}, \phi_{g\mathscr{P}}).$$

Since we canonically have $\mathcal{G} \times^{\mathcal{G}^\circ} \mathscr{P} \cong \mathcal{G} \times^{\mathcal{G}^\circ} g\mathscr{P}$, the construction $\mathscr{P} \mapsto \mathcal{G} \times^{\mathcal{G}^\circ} \mathscr{P}$ naturally induces a map

$$[\mathrm{Sht}_{\mathcal{G}^\circ,\mu}/\pi_0(\mathcal{G})^\phi] \to \mathrm{Sht}_{\mathcal{G},\mu}.$$

3.3. **Decomposition of the moduli stack of quasi-parahoric shtukas.** Our goal in this section is to prove Theorem III. We once again assume $\mathcal{G}/\mathbb{Z}_p$ is a quasi-parahoric group scheme with associated parahoric $\mathcal{G}^\circ$. As in 2.2.9, we denote by $\Pi_{\mathcal{G}}$ the kernel of $H^1(\mathbb{Z}_p, \mathcal{G}) \to H^1(\mathbb{Q}_p, \mathcal{G})$.

3.3.1. For each $\delta \in \Pi_{\mathcal{G}}$, fix a choice of $\dot{\gamma} \in N(\breve{\mathbb{Q}}_p)$ as in Section 2.2.9. We now construct maps

$$\mathrm{Sht}_{\mathcal{G}_\delta,\mu,\mathcal{O}_{\breve{E}}} \to \mathrm{Sht}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}$$

following [PR22, Section 4.4]. Suppose we are given a $\mathcal{G}_\delta$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ over $\mathrm{Spa}(R, R^+)$, where the leg is over an untilt $\mathcal{O}_{\breve{E}} \to R^{\sharp+}$. Note that R is canonically an $\overline{\mathbb{F}}_p$-algebra, and hence $\mathrm{Spa}(R, R^+) \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ naturally lives over $\mathrm{Spa}(\breve{\mathbb{Z}}_p)$. There is an isomorphism of group schemes

$$\mathrm{Int}\,\dot{\gamma}^{-1}\colon \mathcal{G}_{\delta,\breve{\mathbb{Z}}_p} \to \mathcal{G}_{\breve{\mathbb{Z}}_p};\quad g \mapsto \dot{\gamma}^{-1}g\dot{\gamma},$$

and hence we can push out the $\mathcal{G}_\delta$-torsor $\mathscr{P}$ to a $\mathcal{G}$-torsor

$$\mathscr{P}_{\dot{\gamma}} = \mathcal{G}_{\breve{\mathbb{Z}}_p} \times^{\mathcal{G}_{\delta,\breve{\mathbb{Z}}_p}} \mathscr{P}.$$

This is equivalent to the description in [PR22, Section 4.4], which is phrased in terms of twisting the $\mathcal{G}$-action.

3.3.2. We now construct the Frobenius action on $\mathscr{P}_{\dot\gamma}$. For $g \in \mathcal{G}_\delta(\breve{\mathbb{Z}}_p)$ we have

$$\phi(\dot\gamma^{-1} g \dot\gamma) = \phi(\dot\gamma^{-1})\phi(g)\phi(\dot\gamma) = \dot\delta(\dot\gamma^{-1}\phi(g)\dot\gamma)\dot\delta^{-1},$$

where $\dot\delta := \phi(\dot\gamma)^{-1}\dot\gamma \in \mathcal{G}(\breve{\mathbb{Z}}_p)$, and hence the diagram

$$\begin{array}{ccccc} \mathrm{Frob}^*_{\breve{\mathbb{Z}}_p}\mathcal{G}_{\delta,\breve{\mathbb{Z}}_p} & & \xrightarrow{\phi_{\mathcal{G}_\delta}} & & \mathcal{G}_{\delta,\breve{\mathbb{Z}}_p} \\ \Big\downarrow{\scriptstyle \mathrm{Frob}^*_{\breve{\mathbb{Z}}_p}\mathrm{Int}\,\dot\gamma^{-1}} & & & & \Big\downarrow{\scriptstyle \mathrm{Int}\,\dot\gamma^{-1}} \\ \mathrm{Frob}^*_{\breve{\mathbb{Z}}_p}\mathcal{G}_{\breve{\mathbb{Z}}_p} & \xrightarrow{\phi_{\mathcal{G}}} & \mathcal{G}_{\breve{\mathbb{Z}}_p} & \xleftarrow{\mathrm{Int}\,\dot\delta} & \mathcal{G}_{\breve{\mathbb{Z}}_p} \end{array}$$

commutes. Writing $S = \mathrm{Spa}(R, R^+)$ as usual, since $\mathrm{Frob}^*_S\mathscr{P}_{\dot\gamma}$ is the pushforward of $\mathrm{Frob}^*_S\mathscr{P}$ along $\mathrm{Frob}^*_{\breve{\mathbb{Z}}_p}\mathrm{Int}_{\dot\gamma^{-1}}$, it follows from the diagram above that $\mathrm{Frob}^*_S\mathscr{P}_{\dot\gamma}$ is isomorphic to the $\mathcal{G}_{\breve{\mathbb{Z}}_p}$-torsor

$$\mathcal{G}_{\breve{\mathbb{Z}}_p} \times^{\mathrm{Int}\,\dot\delta, \mathcal{G}_{\breve{\mathbb{Z}}_p}} (\mathcal{G}_{\breve{\mathbb{Z}}_p} \times^{\mathcal{G}_{\delta,\breve{\mathbb{Z}}_p}} \mathrm{Frob}^*_S\mathscr{P}).$$

Therefore, using $\phi_{\mathscr{P}}$, we may construct the meromorphic map

$$\phi_{\mathscr{P}_{\dot\gamma}} \colon \mathrm{Frob}^*_S\mathscr{P}_{\dot\gamma} = \mathcal{G}_{\breve{\mathbb{Z}}_p} \times^{\mathrm{Int}\,\dot\delta, \mathcal{G}_{\breve{\mathbb{Z}}_p}} (\mathcal{G}_{\breve{\mathbb{Z}}_p} \times^{\mathcal{G}_{\delta,\breve{\mathbb{Z}}_p}} \mathrm{Frob}^*_S\mathscr{P}) \xrightarrow{(\mathrm{Int}\,\dot\delta^{-1},\ \phi_{\mathscr{P}})} \mathscr{P}_{\dot\gamma}.$$

Proposition 3.3.3. *For $S \in \mathrm{Perf}$ and $(\mathscr{P}, \phi_{\mathscr{P}})$ an S-point of $\mathrm{Sht}_{\mathcal{G}_\delta,\mu,\mathcal{O}_{\breve{E}}}$, the induced shtuka $(\mathscr{P}_{\dot\gamma}, \phi_{\mathscr{P}_{\dot\gamma}})$ defines an S-point of $\mathrm{Sht}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}$.*

Proof. As in the proof of Proposition 3.2.6, this follows from the fact that conjugation by $\dot\gamma^{-1}$ induces an isomorphism between the local models $\mathbb{M}^{\mathrm{v}}_{\mathcal{G}_\delta,\mu,\mathcal{O}_{\breve{E}}}$ and $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}$, together with the fact that $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}$ is stable under conjugation by $\dot\delta^{-1}$. □

3.3.4. By combining Proposition 3.2.6, Proposition 3.3.3, and Lemma 3.1.9, we obtain a map

$$[\mathrm{Sht}_{\mathcal{G}^\circ_\delta,\mu,\mathcal{O}_{\breve{E}}}/\pi_0(\mathcal{G}_\delta)^\phi] \to \mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}$$

for each $\delta \in \Pi_{\mathcal{G}}$. We now have the following key result.

Theorem 3.3.5. *The map*

$$\coprod_{\delta\in\Pi_{\mathcal{G}}} [\mathrm{Sht}_{\mathcal{G}^\circ_\delta,\mu,\mathcal{O}_{\breve{E}}}/\pi_0(\mathcal{G}_\delta)^\phi] \to \mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}$$

is an isomorphism.

The strategy of the proof is to reduce to the statement for rank-one geometric points, and then verify the isomorphism on each Newton stratum using Proposition 3.1.10 and [PR22, Proposition 4.3.4].

Lemma 3.3.6. *For every quasi-parahoric group $\mathcal{G}/\mathbb{Z}_p$ and geometric conjugacy class of a cocharacters μ with reflex field E, the map $\mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Spd}(\mathcal{O}_E)$ is proper*.*

Proof. As noted after [PR24, Lemma 2.4.4], the exact tensor category of vector bundle shtukas on $\mathrm{Spa}(R, R^+)$ agrees with that of $\mathrm{Spa}(R, R^\circ)$, and therefore the map $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Spd}(\mathbb{Z}_p)$ has uniquely existing lifts along

$$s = \mathrm{Spa}((\textstyle\prod_i C_i^+)[\varpi^{-1}], \prod_i \mathcal{O}_{C_i}) \to \mathrm{Spa}((\prod_i C_i^+)[\varpi^{-1}], \prod_i C_i^+) = S.$$

On the other hand, $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \hookrightarrow \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}(\mathcal{O}_E)}$ is a closed immersion, so the map $\mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Spd}(\mathcal{O}_E)$ also has uniquely existing lifts along $s \to S$ by Lemma 2.1.10.

We now produce uniquely existing lifts along

$$\textstyle\coprod_i s_i = \coprod_i \mathrm{Spa}(C_i, \mathcal{O}_{C_i}) \to \mathrm{Spa}((\prod_i \mathcal{O}_{C_i})[\varpi^{-1}], \prod_i \mathcal{O}_{C_i}) = s,$$

following the argument of [Zha23, Proposition 11.10]. Using [GI23, Proposition 9.5], [Ked20, Theorem 3.8], and [PR22, Proposition 3.2.2], we see that an s-point of $\mathrm{Sht}_{\mathcal{G}}$ corresponds to a $\mathcal{G}$-torsor $\mathscr{P}$ on $\mathrm{Spec}(W(\prod_i \mathcal{O}_{C_i}))$ together with a meromorphic map $\phi_{\mathscr{P}} \colon \mathrm{Frob}^*\mathscr{P} \dashrightarrow \mathscr{P}$, and similarly for each s_i-point. Using the Tannakian formalism and that $W(\mathcal{O}_{C_i})$ are local rings, we first observe that the groupoid of $\mathcal{G}$-torsors over $W(\prod_i \mathcal{O}_{C_i}) = \prod_i W(\mathcal{O}_{C_i})$ is canonically equivalent to the product of the groupoids of $\mathcal{G}$-torsors over $W(\mathcal{O}_{C_i})$. Next, to control the meromorphic Frobenius action, we use the fact that $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \to \mathrm{Spd}(\mathcal{O}_E)$ is proper*, which follows from it being proper and representable, together with Lemma 2.1.10. By trivializing the $\mathcal{G}$-torsors, this implies that given a collection of $\mathcal{G}$-torsors on each $W(\mathcal{O}_{C_i})$ with meromorphic Frobenius actions bounded by μ, their product is a $\mathcal{G}$-torsor on $\prod_i W(\mathcal{O}_{C_i})$ with meromorphic Frobenius action again bounded by μ.

□

Proof of Theorem 3.3.5. We first check that the map is proper*. By Lemma 2.1.9, it suffices to show that the structure maps $[\mathrm{Sht}_{\mathcal{G}^\circ_\delta,\mu,\mathcal{O}_{\breve{E}}}/\pi_0(\mathcal{G}_\delta)^\phi] \to \mathrm{Spd}(\mathcal{O}_{\breve{E}})$ and $\mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}} \to \mathrm{Spd}(\mathcal{O}_{\breve{E}})$ are proper*. This follows by combining Lemma 2.1.11, Lemma 2.1.9, and Lemma 3.3.6.

At this point, it suffices to show that for every algebraically closed perfectoid field C the map

$$\coprod_{\delta\in\Pi_{\mathcal{G}}} [\mathrm{Sht}_{\mathcal{G}^\circ_\delta,\mu,\mathcal{O}_{\breve{E}}}/\pi_0(\mathcal{G}_\delta)^\phi](C) \to \mathrm{Sht}^{\kappa=-\mu^\natural}_{\mathcal{G},\mu,\mathcal{O}_{\breve{E}}}(C)$$

is an equivalence of groupoids. We can verify this one Newton stratum at a time, and by Proposition 3.1.10 and Lemma 3.1.9, we only need to work with Newton strata corresponding to elements in $B(G, \mu^{-1})$. For $[b] \in B(G, \mu^{-1})$ choose $b \in G(\breve{\mathbb{Q}}_p)$ with $b \in [b]$. Then by Lemma 3.1.6, we may identify the restriction of the map in the statement of Theorem 3.3.5 to the Newton stratum corresponding to $[b]$, with the map

$$\coprod_{\delta\in\Pi_{\mathcal{G}}} \left[\left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ_\delta,b,\mu}/\pi_0(\mathcal{G}_\delta)^\phi\right)/\tilde{G}_b\right] \to \left[\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}/\tilde{G}_b\right].$$

By construction, the induced map

$$\coprod_{\delta\in\Pi_{\mathcal{G}}} \mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ_\delta,b,\mu}/\pi_0(\mathcal{G}_\delta)^\phi \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$$

agrees with the one constructed by Pappas and Rapoport in [PR22, Equation (4.4.1)], which by Theorem 4.4.1 of *loc. cit.* is an isomorphism. Thus the natural map in 3.3.5 is a bijection on rank one geometric points, and by Lemma 3.3.6 it is also a bijection on products of geometric points. Since both sides are v-stacks, while products of points form a basis of the v-topology by [Gle20, Remark 1.3], we are done. □

Corollary 3.3.7. *The natural map*

$$\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$$

is a torsor for the abelian group $\pi_0(\mathcal{G}_\delta)^\phi$.

Proof. By Theorem 3.3.5, the map is finite étale upon base changing along $\mathrm{Spd}(\mathcal{O}_{\breve{E}}) \to \mathrm{Spd}(\mathcal{O}_E)$. Since $\mathrm{Spd}(\mathcal{O}_{\breve{E}}) \to \mathrm{Spd}(\mathcal{O}_E)$ is v-surjective, the map $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu}$ is also finite étale according to [Sch17, Corollary 9.11]. □

3.3.8. It follows that the image of the map $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu}$ is an open and closed substack. We will denote this image by $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ so that

$$\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$$

is an étale torsor for the finite group $\pi_0(\mathcal{G})^\phi$.

Corollary 3.3.9. *There is a natural isomorphism*

$$\mathrm{Sht}_{\mathcal{G},\mu,\delta=1} \times_{\mathrm{Spd}(\mathcal{O}_E)} \mathrm{Spd}(E) \simeq \left[\mathrm{Gr}_{G,\mu^{-1}}/\underline{\mathcal{G}(\mathbb{Z}_p)}\right].$$

Proof. This is true for $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$ by [Zha23, Proposition 11.16], and the result now follows from the short exact sequence

$$1 \to \mathcal{G}^\circ(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{Z}_p) \to \pi_0(\mathcal{G})^\phi \to 1.$$

and Corollary 3.3.9. □

3.3.10. Now let $\mathcal{H}$ be another quasi-parahoric model of G such that $\mathcal{G}^\circ \subset \mathcal{H} \subset \mathcal{G}$. Then we have the following corollary.

Corollary 3.3.11. *The natural map* $\mathrm{Sht}_{\mathcal{H},\mu,\delta=1} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ *is a torsor for the finite abelian group* $\pi_0(\mathcal{G})^\phi/\pi_0(\mathcal{H})^\phi$.

Proof. Let $\pi_0(\mathcal{H}) \subset \pi_0(\mathcal{G})$ be the inclusion induced by $\mathcal{H} \subset \mathcal{G}$. Then applying the discussion in Section 3.3.8 to both $\mathcal{H}$ and $\mathcal{G}$, we can identify the map in concern with the natural map

$$\left[\mathrm{Sht}_{\mathcal{G}^\circ,\mu}/\pi_0(\mathcal{H})^\phi\right] \to \left[\mathrm{Sht}_{\mathcal{G}^\circ,\mu}/\pi_0(\mathcal{G})^\phi\right],$$

which is clearly a torsor for $\pi_0(\mathcal{G})^\phi/\pi_0(\mathcal{H})^\phi$. □

Remark 3.3.12. The subgroup $\mathcal{H} \subset \mathcal{G}$ is *not* determined by the subgroup $\mathcal{H}(\mathbb{Z}_p) \subset \mathcal{G}(\mathbb{Z}_p)$, because the latter only depends on $\pi_0(\mathcal{H})^\phi$ and not on $\pi_0(\mathcal{H})$ itself. Nevertheless, Corollary 3.3.11 tells us that the stack $\mathrm{Sht}_{\mathcal{H},\mu,\delta=1}$ only depends on $\mathcal{H}(\mathbb{Z}_p)$.

Indeed, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two quasi-parahoric models of G such that $\mathcal{H}_1(\mathbb{Z}_p) = \mathcal{H}_2(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$, then

$$\mathcal{H}_1^\circ(\mathbb{Z}_p) = \mathcal{H}_1(\mathbb{Z}_p) \cap G(\mathbb{Q}_p)^0 = \mathcal{H}_2(\mathbb{Z}_p) \cap G(\mathbb{Q}_p)^0 = \mathcal{H}_2^\circ(\mathbb{Z}_p).$$

Thus the identity components $\mathcal{H}_1^\circ$ and $\mathcal{H}_2^\circ$ are parahoric integral models of G with the same $\mathbb{Z}_p$-points, and therefore they must be isomorphic. Since $\mathcal{H}_1(\mathbb{Z}_p) = \mathcal{H}_2(\mathbb{Z}_p)$ we moreover find that $\pi_0(\mathcal{H}_1)^\phi = \pi_0(\mathcal{H}_2)^\phi$ as subgroups of $\pi_1(G)_{I_p}^\phi$. Corollary 3.3.11 and its proof now tell us that there is an isomorphism

$$\mathrm{Sht}_{\mathcal{H}_1,\mu,\delta=1} \simeq \mathrm{Sht}_{\mathcal{H}_2,\mu,\delta=1}\,.$$

4. Conjectural canonical integral models

Let (G, X) be a Shimura datum with reflex field E, let p be a prime and write $G = \mathsf{G}_{\mathbb{Q}_p}$. Let $\mathcal{G}$ be a quasi-parahoric model of G over $\mathbb{Z}_p$, and let $K_p = \mathcal{G}(\mathbb{Z}_p)$. Choose a prime v of E above p, and let E denote the completion of E at v. Let μ denote the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of G corresponding to X and v. We will write $\mathcal{O}_E$ for the ring of integers of E and k_E for its residue field. For $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ a sufficiently small compact open subgroup we write $K = K_pK^p$. Associated to (G, X) and K^p is the Shimura variety $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$, which we view as an E-scheme (i.e., we take the base change to E of the canonical model over E).

We will often consider Shimura varieties with infinite level structures. In particular, we let

$$\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X}) = \varprojlim_{K_p' \subset K_p} \mathbf{Sh}_{K_p'K^p}(\mathsf{G}, \mathsf{X}) \tag{4.0.1}$$

as $K_p' \subset K_p$ varies over all compact open subgroups of the fixed K_p, and let

$$\mathbf{Sh}_{K_p}(\mathsf{G}, \mathsf{X}) = \varprojlim_{K^p \subset \mathsf{G}(\mathbb{A}_f^p)} \mathbf{Sh}_{K_pK^p}(\mathsf{G}, \mathsf{X})$$

as K^p varies over all sufficiently small compact open subgroups $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$.

Let Z° denote the connected component of the center of G. We will assume that (G, X) satisfies

$$\mathrm{rank}_{\mathbb{Q}}(\mathsf{Z}^\circ) = \mathrm{rank}_{\mathbb{R}}(\mathsf{Z}^\circ). \tag{4.0.2}$$

This equality is equivalent to Milne's axiom SV5 [Mil05, p.63] by [KSZ21, Lemma 1.5.5].

Remark 4.0.1. By [KSZ21, Lemma 5.1.2.(i)], the assumption (4.0.2) is satisfied whenever (G, X) is of Hodge type, which will be the main case of interest to us.

4.1. **Canonical integral models, after Pappas–Rapoport.**

4.1.1. *Shtukas.* Each finite level Shimura variety $\mathbf{Sh}_{K'_pK^p}(\mathsf{G},\mathsf{X})$ is a smooth algebraic variety over E, and the transition maps in the tower (4.0.1) are finite étale. We denote by $\mathbb{P}_K$ the pro-étale $\mathcal{G}(\mathbb{Z}_p)$-cover

$$\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_K(\mathsf{G},\mathsf{X}).$$

Let μ denote the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of G coming from the Hodge cocharacter and the place v. There is a $G(\mathbb{Q}_p)$-equivariant Hodge–Tate period map $\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^\lozenge \to \mathrm{Gr}_{G,\mu^{-1}}$, see [PR24, Proposition 4.1.2] or [Rod22, Corollary 4.1.5]. Thus we have a map $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^\lozenge \to \left[\mathrm{Gr}_{G,\mu^{-1}}/\mathcal{G}(\mathbb{Z}_p)\right]$, which by Corollary 3.3.9 gives us a map

$$\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^\lozenge \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1} \subset \mathrm{Sht}_{\mathcal{G},\mu}.$$

We denote the corresponding $\mathcal{G}$-shtuka by $\mathscr{P}_{K,E}$. Inspired by the axioms in [PR24, Conjecture 4.2.2], we make the following definition.

Definition 4.1.2. Let $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p\subset\mathsf{G}(\mathbb{A}_f^p)}$ be a system of normal schemes that are flat, separated and of finite-type over $\mathcal{O}_E$, with generic fiber $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$, and with K^p varying over all sufficiently small compact open subgroups of $\mathsf{G}(\mathbb{A}_f^p)$. We say the system $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ is a *canonical integral model for* $\{\mathbf{Sh}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ if the following properties are satisfied:

(i) For every discrete valuation ring R of characteristic $(0,p)$ over $\mathcal{O}_E$,

$$\mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})(R[1/p]) = \left(\varprojlim_{K^p} \mathscr{S}_K(\mathsf{G},\mathsf{X})\right)(R).$$

(ii) For every $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$, $g\in\mathsf{G}(\mathbb{A}_f^p)$, and K'^p with $gK'^pg^{-1}\subset K^p$, there are finite étale morphisms $[g] : \mathscr{S}_{K'}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_K(\mathsf{G},\mathsf{X})$ extending the natural maps on the generic fiber.

(iii) The $\mathcal{G}$-shtuka $\mathscr{P}_{K,E}$ on $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^\lozenge$ extends to a $\mathcal{G}$-shtuka $\mathscr{P}_K$ on $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\lozenge/}$ for every sufficiently small $K^p\subset\mathsf{G}(\mathbb{A}_f^p)$.

(iv) Let ℓ be an algebraically closed field of characteristic p together with an embedding $e\colon k_E \hookrightarrow \ell$. For $x\in\mathscr{S}_K(\mathsf{G},\mathsf{X})(\ell)$ with corresponding $b_x \in \mathrm{Sht}_{\mathcal{G},\mu}(\mathrm{Spd}(\ell))$, let $x_0\in\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu}(\mathrm{Spd}(\ell))$ be the base point as in Remark 3.1.7. Then there is an isomorphism of completions

$$\Theta_x : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu/x_0}} \xrightarrow{\sim} (\widehat{\mathscr{S}_K(\mathsf{G},\mathsf{X})_{W_{\mathcal{O}_E,e}(\ell),/x}})^\lozenge,$$

under which the shtuka $\Theta_x^*(\mathscr{P}_K)$ agrees with the universal shtuka $\mathscr{P}^{\mathrm{univ}}$ on $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu}$ coming from the map $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}\to\mathrm{Sht}_{\mathcal{G},\mu}$ of Lemma 3.1.6. Here the left hand side is defined as in [Gle20, Definition 4.18], see the explanation in [PR22, Section 3.3.1-2].

Remark 4.1.3. The extension of the $\mathcal{G}$-shtuka in (iii) is necessarily unique up to unique isomorphism. As in the proof of [PR24, Corollary 2.7.10], even for quasi-parahoric groups $\mathcal{G}$ we can use the Tannakian formalism to reduce to the [PR24, Theorem 2.7.7].

The following conjecture is an extension of [PR24, Conjecture 4.2.2] to the case of quasi-parahoric $\mathcal{G}$.

Conjecture 4.1.4. *For every Shimura datum* (G, X) *satisfying* (4.0.2) *and* $\mathcal{G}/\mathbb{Z}_p$ *a quasi-parahoric model of* G*, if we set* $K_p = \mathcal{G}(\mathbb{Z}_p)$*, then there exists a system of canonical integral models* $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$ *of* $\{\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$.

By [PR24, Theorem 4.5.2], a system of canonical integral models exists in the case of a Hodge-type Shimura datum under the additional assumption that $\mathcal{G}$ is a stabilizer parahoric (this notion was introduced in Section **??**). The conjecture is also known to hold if (G, X) is of toral type (i.e., if $\mathsf{G} = \mathsf{T}$ is a torus) and $\mathcal{G}$ is parahoric, by [Dan22, Theorem A].[13] We will show in Section 4.2, see Theorem 4.2.3, that the conjecture holds for all Hodge type Shimura data (G, X) and all quasi-parahoric models $\mathcal{G}$.

Remark 4.1.5. The map $\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\Diamond/} \to \mathrm{Sht}_{\mathcal{G},\mu}$ automatically factors through $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ if it exists. Indeed, the inclusion $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1} \to \mathrm{Sht}_{\mathcal{G},\mu}$ is open and closed and the factorization property is true when restricted to $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\Diamond} \subset \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\Diamond/}$ by construction. We now conclude using the fact that the inclusion $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\Diamond} \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\Diamond/}$ induces a surjection on π_0. Indeed, taking π_0 of the pushout diagram

$$\begin{array}{ccc} (\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond})_E & \longrightarrow & \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamond} \\ \downarrow & & \downarrow \\ \mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\Diamond} & \longrightarrow & \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\Diamond/} \end{array}$$

gives a pushout diagram of π_0's. But we know that the top arrow is surjective on connected components by flatness of $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$ and [AGLR22, Lemma 2.17]. This implies the desired surjectivity on π_0 for the bottom arrow. This also shows that the basepoint x_0 lies in the image of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ, b_x, \mu, e} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G}, b_x, \mu, e}$.

Remark 4.1.6. Recall from Remark 3.3.12 that quasi-parahoric models $\mathcal{G}$ are typically not determined by their set of $\mathbb{Z}_p$-points. Thus a priori it is possible that one could use different quasi-parahoric models $\mathcal{G}$ with the same $\mathbb{Z}_p$-points to give rise to different axioms for integral models of the Shimura variety of level $\mathcal{G}(\mathbb{Z}_p)$. However, by Remark 3.3.12 and Remark 4.1.5, this does not happen.

4.1.7. Let $\iota : (\mathsf{G}, \mathsf{X}) \to (\mathsf{G}', \mathsf{X}')$ be a closed embedding of Shimura data. Write E, E' for the corresponding reflex fields. Choose a place v of E above p and let v' be the induced place of $\mathsf{E}' \subset \mathsf{E}$; we let $E' \subset E$ denote the induced map on completions. Let $\mathcal{G}$ and $\mathcal{G}'$ be quasi-parahoric models of G and G' respectively; write $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $U_p = \mathcal{G}'(\mathbb{Z}_p)$. We assume that $K_p = \iota^{-1}(U_p) \cap G(\mathbb{Q}_p)$. For every sufficiently small compact open subgroup $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ we choose $U^p \subset \mathsf{G}'(\mathbb{A}_f^p)$ such that ι induces a

[13] In fact, an extension of Conjecture 4.1.4 is proven in [Dan22] for (G, X) of toral type which do not necessarily satisfy (4.0.2). In this case one needs to work with a variant $\mathcal{G}^c$ of $\mathcal{G}$; see [Dan22, Section 4.2 and Section 4.3] for details.

closed immersion (see [Kis10, Lemma 2.1.2])

$$\mathbf{Sh}_K(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_U(\mathsf{G}',\mathsf{X}') \times_{\mathrm{Spec}(E)'} \mathrm{Spec}(E), \tag{4.1.1}$$

where $U = U^pU_p$ and $K = K^pK_p$. We have the following version of [PR24, Theorem 4.3.1, Theorem 4.5.2].

Theorem 4.1.8 (Pappas–Rapoport). *Let $\{\mathscr{S}_U(\mathsf{G}',\mathsf{X}')\}_{U^p}$ be a canonical integral model of $\{\mathbf{Sh}_U(\mathsf{G}',\mathsf{X}')\}_{U^p}$. If $\mathcal{G}(\breve{\mathbb{Z}}_p) = \iota^{-1}(\mathcal{G}'(\breve{\mathbb{Z}}_p)) \cap G(\breve{\mathbb{Q}}_p)$, then there is a canonical integral model $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ of $\{\mathbf{Sh}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ such that the morphism in* (4.1.1) *extends uniquely to a morphism*

$$\iota : \mathscr{S}_K(\mathsf{G},\mathsf{X}) \to \mathscr{S}_U(\mathsf{G}',\mathsf{X}') \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E$$

over $\mathrm{Spec}(\mathcal{O}_E)$*, such that the following diagram commutes*

$$\begin{array}{ccc} \mathscr{S}_K(\mathsf{G},\mathsf{X})^{\Diamond/} & \xrightarrow{\pi_{\mathrm{crys},\mathcal{G}}} & \mathrm{Sht}_{\mathcal{G},\mu} \\ \downarrow{\scriptstyle \iota} & & \downarrow \\ \mathscr{S}_U(\mathsf{G}',\mathsf{X}')^{\Diamond/} \times_{\mathrm{Spd}(\mathcal{O}_{E'})} \mathrm{Spd}(\mathcal{O}_E) & \xrightarrow{\pi_{\mathrm{crys},\mathcal{G}'}} & \mathrm{Sht}_{\mathcal{G}',\mu'} \times_{\mathrm{Spd}(\mathcal{O}_{E'})} \mathrm{Spd}(\mathcal{O}_E). \end{array}$$

Proof. This follows as in the proofs of [PR24, Theorem 4.3.1, Theorem 4.5.2], with some small modifications as outlined below.

We define $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ to be the normalization of the Zariski closure of $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E$ in $\mathscr{S}_U(\mathsf{G}',\mathsf{X}')$ for all K^p. Axioms (i) and (ii) follow as in the proofs of [PR24, Theorem 4.5.2]. It remains to show that $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ satisfies axioms (iii) and (iv).

The assumption that $\mathcal{G}(\breve{\mathbb{Z}}_p) = \iota^{-1}(\mathcal{G}'(\breve{\mathbb{Z}}_p)) \cap G(\breve{\mathbb{Q}}_p)$ implies that there is a natural map $\mathcal{G} \to \mathcal{G}'$ extending $G \to G'$ on the generic fiber, see [KP23, Corollary 2.10.10], which identifies $\mathcal{G}$ with the group smoothening of the Zariski closure $\overline{\mathcal{G}}$ of G in $\mathcal{G}'$; this is explained in [PR24, Section 4.3].

Thus we obtain a commutative diagram (from now on we implicitly base change $\mathrm{Sht}_{\mathcal{G}',\mu'}$ to $\mathrm{Spd}(\mathcal{O}_E)$)

$$\begin{array}{ccc} \mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E^{\Diamond} & \longrightarrow & \mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}(\mathcal{O}_E)} \mathrm{Spd}(E) \\ \downarrow & & \downarrow \\ \mathbf{Sh}_U(\mathsf{G}',\mathsf{X}')_E^{\Diamond} & \longrightarrow & \mathrm{Sht}_{\mathcal{G}',\mu'} \times_{\mathrm{Spd}(\mathcal{O}_E)} \mathrm{Spd}(E). \end{array}$$

By assumption the bottom horizontal arrow extends (necessarily uniquely) to a morphism $\mathscr{S}_{U^pU_p}(\mathsf{G}',\mathsf{X}')^{\Diamond/} \to \mathrm{Sht}_{\mathcal{G}',\mu'}$. We want to show that the top horizontal arrow extends (necessarily uniquely) to a morphism

$$\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\Diamond/} \to \mathrm{Sht}_{\mathcal{G},\mu}$$

for all sufficiently small K^p. The existence of this extension of the $\mathcal{G}$-shtuka can be proved by following the argument in [PR24, Section 4.6], taking into account the modifications to these arguments discussed in [PR24, Section 4.8.1]. Moreover we should take into account that [Ans22, Corollary 11.6], used in [PR24, Lemma

4.6.6], has been extended to include stabilizer Bruhat–Tits group schemes, see [PR22, Proposition 3.2.1, Proposition 3.2.2].

We observe that the proof of [PR24, Proposition 4.7.1] goes through for quasi-parahoric $\mathcal{G}$ and with k_E replaced by an arbitrary algebraically closed field ℓ. The proof of Axiom (iv) in [PR24, Section 4.7.1] then applies to prove axiom (iv) for $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\lozenge/}$. □

We have the following version of [PR24, Theorem 4.2.4].

Corollary 4.1.9. *A canonical integral model* $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ *of* $\{\mathbf{Sh}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ *is unique up to unique isomorphism, if it exists.*

Proof. Let $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}, \{\mathscr{S}'_K(\mathsf{G},\mathsf{X})\}_{K^p}$ be two canonical integral models. Then their product $\{\mathscr{S}_K(\mathsf{G},\mathsf{X}) \times \mathscr{S}'_K(\mathsf{G},\mathsf{X})\}_{K^p \times K^p}$, equipped with the obvious $\mathcal{G} \times \mathcal{G}$-shtuka, defines a canonical integral model of $\{\mathbf{Sh}_U(\mathsf{G} \times \mathsf{G}, \mathsf{X} \times \mathsf{X})\}_{U^p}$ (using the fact that compact open subgroups U^p of the form $K^p \times K^p$ are cofinal). If we apply Theorem 4.1.8 to the diagonal morphism $(\mathsf{G},\mathsf{X},\mathcal{G}) \to (\mathsf{G} \times \mathsf{G}, \mathsf{X} \times \mathsf{X}, \mathcal{G} \times \mathcal{G})$ with respect to the integral model $\{\mathscr{S}_K(\mathsf{G},\mathsf{X}) \times \mathscr{S}'_K(\mathsf{G},\mathsf{X})\}_{K^p \times K^p}$, then we get a third integral model $\{\mathscr{S}''_K(\mathsf{G},\mathsf{X})\}_{K^p}$ together with a morphism

$$\mathscr{S}''_K(\mathsf{G},\mathsf{X}) \to \mathscr{S}_K(\mathsf{G},\mathsf{X}) \times \mathscr{S}'_K(\mathsf{G},\mathsf{X})$$

extending the diagonal map on the generic fiber. It can be shown as in the proof of [PR24, Theorem 4.2.4] that this map is the graph of an isomorphism. □

We have the following version of [PR24, Corollary 4.3.2]. The result there is only stated for the change-of-parahoric maps for a fixed (G,X), but the proof works for all morphisms of triples $(\mathsf{G},\mathsf{X},\mathcal{G})$.

Corollary 4.1.10. *Let* $f\colon (\mathsf{G},\mathsf{X},\mathcal{G}) \to (\mathsf{G}',\mathsf{X}',\mathcal{G}')$ *be a morphism of triples admitting canonical integral models* $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ *and* $\{\mathscr{S}_U(\mathsf{G}',\mathsf{X}')\}_{U^p}$. *Then there uniquely exists a morphism of integral models* $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p} \to \{\mathscr{S}_U(\mathsf{G}',\mathsf{X}')\}_{U^p}$ *extending the morphism on generic fibers.*

Proof. We first observe that the triple $(\mathsf{G} \times \mathsf{G}', \mathsf{X} \times \mathsf{X}', \mathcal{G} \times \mathcal{G}')$ admits integral models satisfying Conjecture 4.1.4. Indeed, as in the proof of Corollary 4.1.9, we can take the product of the integral models for $(\mathsf{G},\mathsf{X},\mathcal{G})$ and $(\mathsf{G}',\mathsf{X}',\mathcal{G}')$ equipped with the obvious $\mathcal{G} \times \mathcal{G}'$-shtuka. Then the graph of f defines a morphism $(\mathsf{G},\mathsf{X},\mathcal{G}) \to (\mathsf{G} \times \mathsf{G}', \mathsf{X} \times \mathsf{X}', \mathcal{G} \times \mathcal{G}')$ which satisfies the assumption of Theorem 4.1.8, since f maps $\mathcal{G}(\breve{\mathbb{Z}}_p)$ to $\mathcal{G}'(\breve{\mathbb{Z}}_p)$ by assumption. This defines a morphism of integral models (using Theorem 4.1.8 and Corollary 4.1.9)

$$g : \mathscr{S}_K(\mathsf{G},\mathsf{X}) \to \mathscr{S}_K(\mathsf{G},\mathsf{X}) \times_{\mathrm{Spec}(\mathcal{O}_E)} \left(\mathscr{S}_U(\mathsf{G}',\mathsf{X}') \times_{\mathrm{Spec}(\mathcal{O}_{E'})} \mathrm{Spec}(\mathcal{O}_E)\right)$$

extending the graph of f on the generic fiber. Its composition with projection to $\mathscr{S}_U(\mathsf{G}',\mathsf{X}')$ gives the desired morphism, which is unique by normality. □

4.1.11. The following theorem will be a crucial ingredient in the proof of Theorem I. Let $\mathcal{G}$ be a quasi-parahoric model of G and let $\mathcal{H} \subset \mathcal{G}$ be a quasi-parahoric subgroup (i.e. $\mathcal{H}^\circ = \mathcal{G}^\circ$). Let $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $K'_p = \mathcal{H}(\mathbb{Z}_p)$. For $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ a sufficiently small compact open subgroup write $K = K^p K_p$ and $K' = K^p K'_p$. Assume for each K^p, we have a normal integral model $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$ of $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$ which is flat, separated and of finite-type over $\mathcal{O}_E$. We define

$$\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X}) \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})$$

to be the relative normalization of $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$ in the composition

$$\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X}) \to \mathbf{Sh}_K(\mathsf{G}, \mathsf{X}) \to \mathscr{S}_K(\mathsf{G}, \mathsf{X}).$$

Theorem 4.1.12. *With the above construction, suppose $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$ is a canonical integral model for $\{\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$. Then $\{\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X})\}_{K^p}$ is a canonical integral model for $\{\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X})\}_{K^p}$.*

Proof of Theorem 4.1.12. We start by noting that axioms (i) and (ii) are a straightforward consequence of the corresponding axioms for $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$.

By Remark 4.1.5, the map $\mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamondsuit/} \to \mathrm{Sht}_{\mathcal{G},\mu}$ factors through $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$. The morphism

$$\mathscr{X} := \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamondsuit/} \times_{\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}} \mathrm{Sht}_{\mathcal{H},\mu,\delta=1} \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})^{\diamondsuit/}$$

is a torsor for the finite abelian group $\pi_0(\mathcal{G})^\phi / \pi_0(\mathcal{H})^\phi$ by Corollary 3.3.11. If we base change to $\mathrm{Spd}(E)$ and apply Corollary 3.3.9 twice, we obtain the Cartesian diagram

$$\begin{array}{ccc} \mathscr{X}_E & \longrightarrow & \left[\mathrm{Gr}_{G,\mu^{-1}} / \underline{\mathcal{H}(\mathbb{Z}_p)}\right] \\ \downarrow & & \downarrow \\ \mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\diamondsuit} & \longrightarrow & \left[\mathrm{Gr}_{G,\mu^{-1}} / \underline{\mathcal{G}(\mathbb{Z}_p)}\right]. \end{array}$$

It follows from the construction of the bottom horizontal map, see Section 4.1.1, that this identifies $\mathscr{X}_E \to \mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\diamondsuit}$ with $\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X})^{\diamondsuit} \to \mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\diamondsuit}$. It now follows from Proposition 2.3.1 that $\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X})^{\diamondsuit/}$ is isomorphic to $\mathscr{X}$. This shows that there is a morphism

$$\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X})^{\diamondsuit/} \to \mathrm{Sht}_{\mathcal{H},\mu,\delta=1},$$

proving axiom (iii). Moreover, we see that $\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X}) \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})$ is finite étale.

Axiom (iv) for $\{\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X})\}_{K^p}$ follows from Axiom (iv) for $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$ together with the following observation: By [PR22, Proposition 4.2.1] the natural map

$$\widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ, b_x, \mu}}_{/x_0} \to \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G}, b_x, \mu}}_{/x_0}$$

is an isomorphism, and the same is true for the natural map of formal completions of $\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X})$ and $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$, since $\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X}) \to \mathscr{S}_K(\mathsf{G}, \mathsf{X})$ is finite étale. □

4.2. **Integral models of Shimura varieties of Hodge type.** For a symplectic space (V, ψ) over $\mathbb{Q}$ we write $\mathsf{G}_V = \mathrm{GSp}(V, \psi)$ for the group of symplectic similitudes of (V, ψ) over $\mathbb{Q}$. It admits a Shimura datum $\mathcal{H}_V$ consisting of the union of the Siegel upper and lower half spaces.

Lemma 4.2.1. *Conjecture 4.1.4 holds for any choice of parahoric $\mathcal{G}_V$ of G_V.*

Proof. This is essentially a special case of [PR24, Theorem 4.5.2]. Our formulation of axiom (iv) is stronger, but the same proof works once we observe that the deformation theory for p-divisible groups as in the proof of [PR24, Lemma 4.10.1] works for arbitrary algebraically closed fields. □

4.2.2. *Main results.* Let (G, X) be a Shimura datum of Hodge type with reflex field E, let p be a prime and write $G = \mathsf{G}_{\mathbb{Q}_p}$. Fix a place v above p of the reflex field E, and let E be the completion of E at v with ring of integers $\mathcal{O}_E$ and residue field k_E. Let $\mathcal{H}$ be any quasi-parahoric integral model of G and write $K_p' = \mathcal{H}(\mathbb{Z}_p)$. For any sufficiently small compact open subgroup $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ we will consider the Shimura variety $\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X})$ of level $K' = K^p K_p'$ as a scheme over E. The following is the main result of this paper and verifies Conjecture 4.1.4.

Theorem 4.2.3. *There exists a canonical integral model $\{\mathscr{S}_{K'}(\mathsf{G}, \mathsf{X})\}_{K^p}$ of $\{\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X})\}_{K^p}$.*

Proof. By Corollary 2.2.7, we may choose a stabilizer Bruhat–Tits group scheme $\mathcal{G}$ such that $\mathcal{H}$ is an open subgroup of $\mathcal{G}$; write $K_p = \mathcal{G}(\mathbb{Z}_p)$. It is explained in [KMPS22, Section 1.3.2] that there exists a Hodge embedding $\iota : (\mathsf{G}, \mathsf{X}) \to (\mathsf{G}_V, \mathsf{H}_V)$ and a $\mathbb{Z}_p$-lattice $V_p \subset V_{\mathbb{Q}_p}$ on which ψ is $\mathbb{Z}_p$-valued, such that $\mathcal{G}(\breve{\mathbb{Z}}_p)$ is the stabilizer in $G(\breve{\mathbb{Q}}_p)$ of $V_p \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$. In other words, if $\mathcal{G}_V$ is the parahoric integral model of G_V over $\mathbb{Z}_p$ that is the stabilizer of V_p, then we have $\mathcal{G}(\breve{\mathbb{Z}}_p) = G(\breve{\mathbb{Q}}_p) \cap \iota^{-1}(\mathcal{G}_V(\breve{\mathbb{Z}}_p))$. It now follows from Theorem 4.1.8 and Lemma 4.2.1 that there exists a canonical integral model $\{\mathscr{S}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$ of $\{\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})\}_{K^p}$, and the theorem is now a direct consequence of Theorem 4.1.12. □

Remark 4.2.4. It follows from the proof of Theorem 4.1.12 that the integral models of Theorem 4.2.3 are constructed as relative normalizations, as in [KP18, Section 4.3] and [KP18, Section 5.1.11]. Thus our integral models agree with those constructed in [KP18, Section 4.3] and [KP18, Section 5.1.11].

4.3. **Local model diagrams and a conjecture of Kisin and Pappas.** Let the notation be as in Section 4. In particular, $\mathcal{G}$ is a quasi-parahoric model of G. As in [PR24, Section 4.9.1], we associate to $\mathcal{G}$ the v-sheaf $\mathcal{G}^\diamondsuit$. Explicitly, if $S = \mathrm{Spa}(R, R^+)$ is in Perf, then $\mathcal{G}^\diamondsuit(S)$ consists of pairs $(S^\sharp, g)$, where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp +})$ is an untilt of S and g is an element of $\mathcal{G}(R^\sharp)$.

In *loc. cit.*, Pappas and Rapoport show that for S in Perf and $f : S \to \mathrm{Sht}_{\mathcal{G},\mu}$ there is a $\mathcal{G}^\diamondsuit$-torsor $\tilde{S} \to S$ equipped with a $\mathcal{G}^\diamondsuit$-equivariant map $\tilde{S} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}$[14]. In

[14]Note that since μ is minuscule, the action of the positive loop group $\mathcal{L}^+\mathcal{G}$ on $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}$ factors through $\mathcal{G}^\diamondsuit$. This defines the $\mathcal{G}^\diamondsuit$-action on the v-sheaf local model.

other words, there is a morphism of stacks

$$\mathrm{Sht}_{\mathcal{G},\mu} \to \left[\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}/\mathcal{G}^{\Diamond}\right].$$

By construction, this morphism is functorial in $\mathcal{G}$ in the sense that, given a morphism $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ of quasi-parahoric group schemes, the diagram

$$\begin{array}{ccc} \mathrm{Sht}_{\mathcal{G}_1,\mu} & \longrightarrow & \left[\mathbb{M}^{\mathrm{v}}_{\mathcal{G}_1,\mu}/\mathcal{G}_1^{\Diamond}\right] \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{\mathcal{G}_2,\alpha\circ\mu} & \longrightarrow & \left[\mathbb{M}^{\mathrm{v}}_{\mathcal{G}_2,\alpha\circ\mu}/\mathcal{G}_2^{\Diamond}\right] \end{array} \tag{4.3.1}$$

is 2-commutative. Here the vertical maps are obtained by functoriality of the constructions of $\mathrm{Sht}_{\mathcal{G}}$ and v-sheaf local models.

4.3.1. By [AGLR22, Theorem 1.11], there is a unique (up to unique isomorphism) absolutely weakly normal scheme $\mathbb{M}_{\mathcal{G},\mu}$ which is flat and proper over $\mathcal{O}_E$, with associated v-sheaf isomorphic to $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}$. A canonical integral model $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ is said to have a *scheme-theoretic local model diagram* if for all sufficiently small K^p there is a smooth morphism of algebraic stacks

$$\pi_{\mathrm{dR},\mathcal{G}} \colon \mathscr{S}_K(\mathsf{G},\mathsf{X}) \to [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}],$$

whose generic fiber comes from the canonical model of the standard principal bundle (base-changed to E), see [Mil90, Theorem 4.1], together with a 2-commutative diagram

$$\begin{array}{ccc} \mathscr{S}_K(\mathsf{G},\mathsf{X})^{\Diamond/} & \xrightarrow{\pi_{\mathrm{crys}}} & \mathrm{Sht}_{\mathcal{G},\mu} \\ \Big\downarrow{\scriptstyle \pi^{\Diamond/}_{\mathrm{dR},\mathcal{G}}} & & \Big\downarrow \\ \left[\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}/\mathcal{G}^{\Diamond/}\right] & \longrightarrow & \left[\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}/\mathcal{G}^{\Diamond}\right]. \end{array}$$

Now assume that (G,X) is of Hodge type. Then by Theorem A.3.3, the local model diagrams of [KZ21, Theorem 5.1.10] and [KP18, Theorem 4.2.7] give scheme-theoretic local model diagrams for $(\mathsf{G},\mathsf{X},\mathcal{G})$, where $\mathcal{G}$ is a stabilizer Bruhat–Tits group scheme. We note that these results are stated under some additional assumptions on $(\mathsf{G},\mathsf{X},\mathcal{G})$ and p that we will make explicit in Section 4.3.5.

4.3.2. Let $\mathcal{G}^{\circ} \subset \mathcal{G}$ be the relative identity component and write $K_p^{\circ} = \mathcal{G}^{\circ}(\mathbb{Z}_p)$ and $K_p = \mathcal{G}(\mathbb{Z}_p)$. For $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ a sufficiently small compact open subgroup we write $K = K^pK_p$ and $K^{\circ} = K^pK_p^{\circ}$. Under the assumptions made in [KP18, Theorem 4.2.7.], Kisin and Pappas conjecture in [KP18, Section 4.3.10], that the composition

$$\mathscr{S}_{K^{\circ}}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_K(\mathsf{G},\mathsf{X}) \to [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}]$$

factors through

$$[\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}^{\circ}] \to [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}].$$

The following proposition shows that such factorization exists, whenever a scheme-theoretic local model diagram exists for $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$.

Proposition 4.3.3. *Suppose that $\{\mathscr{S}_K(\mathsf{G},\mathsf{X})\}_{K^p}$ admits a scheme-theoretic local model diagram $\pi_{\mathrm{dR},\mathcal{G}}$. Then $\{\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})\}_{K^p}$ admits a scheme-theoretic local model diagram $\pi_{\mathrm{dR},\mathcal{G}^\circ}$ such that for all (sufficiently small) K^p, the following diagram commutes (here we identify $\mathbb{M}_{\mathcal{G}^\circ,\mu} = \mathbb{M}_{\mathcal{G},\mu}$ via the isomorphism (3.1.1))*

$$\begin{array}{ccc} \mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X}) & \xrightarrow{\pi_{\mathrm{dR},\mathcal{G}^\circ}} & [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}^\circ] \\ \downarrow & & \downarrow \\ \mathscr{S}_K(\mathsf{G},\mathsf{X}) & \xrightarrow{\pi_{\mathrm{dR},\mathcal{G}}} & [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}]\,. \end{array} \tag{4.3.2}$$

To prove the proposition, we will need a lemma. As motivation, we recall from the proof of [PR22, Proposition 3.2.1] that there is a short exact sequence

$$1 \to \mathcal{G}^\circ \to \mathcal{G} \to j_*\pi_0(\mathcal{G}) \to 1 \tag{4.3.3}$$

on the (big) étale site of $S = \mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})$, where we view $\pi_0(\mathcal{G})$ as an étale group scheme over $S_{k_E} := \mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})_{k_E}$, and j is the closed immersion $j : S_{k_E} \hookrightarrow S$.

Lemma 4.3.4. *Let $j : \mathrm{Spd}(\mathbb{F}_p) \to \mathrm{Spd}(\mathbb{Z}_p)$ denote the inclusion. There is a diagram of short exact sequence of v-sheaves of groups over $\mathrm{Spd}(\mathbb{Z}_p)$*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{G}^{\circ,\Diamond/} & \longrightarrow & \mathcal{G}^{\Diamond/} & \longrightarrow & j_*\underline{\pi_0(\mathcal{G})} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathcal{G}^{\circ,\Diamond} & \longrightarrow & \mathcal{G}^{\Diamond} & \longrightarrow & j_*\underline{\pi_0(\mathcal{G})} & \longrightarrow & 1. \end{array}$$

Proof. Note that we can check exactness after base changing to $\mathrm{Spd}(\breve{\mathbb{Z}}_p)$. For surjectivity of $\mathcal{G}^{\Diamond/}_{\breve{\mathbb{Z}}_p} \to (j_*\underline{\pi_0(\mathcal{G})})_{\breve{\mathbb{Z}}_p}$, we observe that there is an open cover $\coprod_{g\in\pi_0(\mathcal{G})(\overline{\mathbb{F}}_p)} \mathrm{Spd}(\breve{\mathbb{Z}}_p) \to (j_*\underline{\pi_0(\mathcal{G})})_{\breve{\mathbb{Z}}_p}$ and a section $\mathrm{Spec}(\breve{\mathbb{Z}}_p) \to \mathcal{G}_{\breve{\mathbb{Z}}_p}$ for each $g \in \pi_0(\mathcal{G})(\overline{\mathbb{F}}_p)$. These induce sections $\mathrm{Spd}(\breve{\mathbb{Z}}_p) \to \mathcal{G}^{\Diamond/}_{\breve{\mathbb{Z}}_p}$, and hence imply surjectivity of $\mathcal{G}^{\Diamond/}_{\breve{\mathbb{Z}}_p} \to (j_*\underline{\pi_0(\mathcal{G})})_{\breve{\mathbb{Z}}_p}$. The kernel of this map can be identified with $\mathcal{G}^{\circ,\Diamond/}_{\breve{\mathbb{Z}}_p}$, because the zero section $\mathrm{Spd}(\breve{\mathbb{Z}}_p) \to (j_*\underline{\pi_0(\mathcal{G})})_{\breve{\mathbb{Z}}_p}$ is an open embedding whose preimage in $\mathcal{G}^{\Diamond/}$ precisely recovers $\mathcal{G}^{\circ,\Diamond/}$. The proof of the exactness of the second row is identical. □

Proof of Proposition 4.3.3. The morphism $\pi_{\mathrm{dR},\mathcal{G}}$ induces a $\mathcal{G}$-torsor $\mathcal{P}'$ over $\mathscr{S}_K(\mathsf{G},\mathsf{X})$; we will denote its pullback to $\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})$ by $\mathcal{P}$. From the short exact sequence (4.3.3), we see that the pushout $\mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G})$ is a torsor for the sheaf of abelian groups $j_*\pi_0(\mathcal{G})$. It suffices to construct a section of it over S. Indeed, given such a section, the pullback along this section of the natural map $\mathcal{P} \to \mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G})$ gives a reduction of $\mathcal{P}$ to a $\mathcal{G}^\circ$-torsor.

By the 2-commutativity of the diagram (4.3.1) applied to $\mathcal{G}^\circ \to \mathcal{G}$, we have a reduction of $\mathcal{P}^\Diamond$ to a $(\mathcal{G}^\circ)^\Diamond$-torsor $\tilde{\mathcal{Q}} \subset \mathcal{P}^\Diamond$. This gives an $S^{\Diamond/}$-point of

$$\mathcal{P}^\Diamond \times^{\mathcal{G}^\Diamond} j_*\underline{\pi_0(\mathcal{G})} \cong \mathcal{P}^{\Diamond/} \times^{\mathcal{G}^{\Diamond/}} j_*\underline{\pi_0(\mathcal{G})} \cong \left(\mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G})\right)^{\Diamond/},$$

where we used Lemma 4.3.4 for the first isomorphism. We want to show that this point is induced by an S-point of $\mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G})$.

We first observe that

$$\mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G}) = j_*\left(\mathcal{G}_{k_E} \times^{\mathcal{P}_{k_E}} \pi_0(\mathcal{G})\right). \tag{4.3.4}$$

From this it follows that

$$H^0\left(S^{\Diamond/}, \left(\mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G})\right)^{\Diamond/}\right) = H^0\left(S^\diamond_{k_E}, \left(\mathcal{P}_{k_E} \times^{\mathcal{G}_{k_E}} \pi_0(\mathcal{G})\right)^\diamond\right).$$

Let $S^{\mathrm{perf}}_{k_E}$ denote the perfection of S_{k_E}. It follows from the full-faithfulness of the functor $X \mapsto X^\diamond$ on perfect schemes, see [SW20, Proposition 18.3.1], that

$$H^0\left(S^\diamond_{k_E}, \left(\mathcal{P}_{k_E} \times^{\mathcal{G}_{k_E}} \pi_0(\mathcal{G})\right)^\diamond\right) = H^0\left(S^{\mathrm{perf}}_{k_E}, \left(\mathcal{P}_{k_E} \times^{\mathcal{G}_{k_E}} \pi_0(\mathcal{G})\right)^{\mathrm{perf}}\right).$$

By topological invariance of the étale site, the right hand side identifies with $H^0(S_{k_E}, \mathcal{P}_{k_E} \times^{\mathcal{G}_{k_E}} \pi_0(\mathcal{G}))$. But from (4.3.4) and the definition of j_*, we have

$$H^0\left(S_{k_E}, \mathcal{P}_{k_E} \times^{\mathcal{G}_{k_E}} \pi_0(\mathcal{G})\right) = H^0\left(S, j_*\left(\mathcal{P}_{k_E} \times^{\mathcal{G}_{k_E}} \pi_0(\mathcal{G})\right)\right).$$

By combining these bijections, we obtain from the $(\mathcal{G}^\circ)^\Diamond$-torsor $\tilde{\mathcal{Q}}$ an S-point of $\mathcal{P} \times^{\mathcal{G}} j_*\pi_0(\mathcal{G})$, i.e., a $\mathcal{G}^\circ$-torsor $\mathcal{Q}$. Clearly $\mathcal{Q}^\Diamond \cong \tilde{\mathcal{Q}}$, since both are the pullback of $\mathcal{P}^\Diamond$ along the same section of $\mathcal{P}^\Diamond \times^{\mathcal{G}^\Diamond} j_*\underline{\pi_0(\mathcal{G})}$ over $S^\Diamond$.

Thus we obtain the desired morphism $\pi_{\mathrm{dR},\mathcal{G}^\circ}$, and it follows from the construction that (4.3.2) commutes. That $\pi_{\mathrm{dR},\mathcal{G}^\circ}$ recovers the canonical model of the standard principle bundle on the generic fiber follows from the corresponding fact for $\pi_{\mathrm{dR},\mathcal{G}}$. Finally, it remains to show that $\pi_{\mathrm{dR},\mathcal{G}^\circ}$ is smooth. This can be checked after pullback to the smooth cover $\mathbb{M}_{\mathcal{G},\mu} \to [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}^\circ]$, but here the map is given by

$$\mathcal{Q} \hookrightarrow \mathcal{P} \xrightarrow{\pi_{\mathrm{dR},\mathcal{G}}|_{\mathcal{P}}} \mathbb{M}_{\mathcal{G},\mu}.$$

While the first map is an open immersion and the second map is smooth by assumption, the composition is smooth. This concludes the proof that $\pi_{\mathrm{dR},\mathcal{G}}$ is a scheme-theoretic local model diagram for $\mathscr{S}_{K^\circ}(\mathsf{G},\mathsf{X})$. □

4.3.5. We now combine Proposition 4.3.3 with Theorem A.3.3 to deduce the existence of scheme-theoretic local model diagrams for Shimura varieties of Hodge type at parahoric level. For this we need the notion of a connected reductive group G over $\mathbb{Q}_p$ being acceptable, [KZ21, Definition 3.3.2], this is automatic if $p \geq 5$ or if G splits over a tamely ramified extension of $\mathbb{Q}_p$. Similarly we need the notion of G being R-smooth, see [DY24, Definition 2.10] and [KZ21, Definition 2.4.3], this is automatic if G splits over a tamely ramified extension of $\mathbb{Q}_p$ by [KZ21, Proposition 2.4.6].

Theorem 4.3.6. *Let $(\mathsf{G}, \mathsf{X}, \mathcal{G}^\circ)$ be a triple of Hodge type with $\mathcal{G}^\circ$ a parahoric. If p is coprime to $2 \cdot \pi_1(G^{\mathrm{der}})$ and the group G is acceptable and R-smooth, then $\mathscr{S}_{K^\circ}(\mathsf{G}, \mathsf{X})$ admits a scheme-theoretic local model diagram.*

Proof. Let us write $\mathcal{G}^\circ$ as the relative identity component of a stabilizer Bruhat–Tits group scheme $\mathcal{G}$. Then under our assumptions Theorem A.3.3 applies, see Remark A.3.1, to produce a scheme-theoretic local model diagram for $\mathscr{S}_K(\mathsf{G}, \mathsf{X})$. The result is now a consequence of Proposition 4.3.3. □

4.4. **Rapoport–Zink uniformization.** Let the notation be as in Section 4.3. In particular, $\mathcal{G}$ is a stabilizer Bruhat–Tits model of G over $\mathbb{Z}_p$ with $K_p = \mathcal{G}(\mathbb{Z}_p)$. Denote by k_E the residue field of $\mathcal{O}_E$ as before. For ℓ an algebraically closed field in characteristic p together with a fixed embedding $e\colon k_E \hookrightarrow \ell$, write

$$W_{\mathcal{O}_E,e}(\ell) = \mathcal{O}_E \otimes_{W(k_E),e} W(\ell)$$

as before. Then for $b \in G(W(\ell)[1/p])$ we have the v-sheaves $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,e}, \mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b,\mu,e}$ and we will also consider the image $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,\delta=1,e}$ of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b,\mu,e} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu,e}$.

Let $x \in \mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})(\ell)$, then its image under π_{crys} defines a $\mathrm{Spd}(\ell)$-point b_x of $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$. Let $e : k_E \to \ell$ be the map corresponding to $x : \mathrm{Spec}(\ell) \to \mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})_{k_E} \to \mathrm{Spec}(k_E)$, then attached to x is a base point

$$x_0\colon \mathrm{Spd}(\ell) \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,e},$$

given by the $\mathrm{Spd}(\ell)$-point of $\mathrm{Sht}_{\mathcal{G},\mu}$ corresponding to $\pi_{\mathrm{crys}}(x)$, see Remark 3.1.7. In fact, since $\pi_{\mathrm{crys}}(x) \in \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$, our base point lies in $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}(\mathrm{Spd}(\ell))$.

Theorem 4.4.1. *If (G, X) is of Hodge type, then there exists a uniformization map*

$$\Theta_{\mathcal{G},x}\colon \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e} \to \mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamond}_{W_{\mathcal{O}_E,e}(\ell)}$$

sending the base point x_0 to x, which restricts to an isomorphism

$$\Theta_{\mathcal{G},x}\colon \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}}_{/x_0} \xrightarrow{\cong} (\widehat{\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})_{W_{\mathcal{O}_E,e}(\ell)}}_{/x})^{\diamond}.$$

Moreover the composition of $\Theta_{\mathcal{G},x}$ with π_{crys}

$$\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e} \to \mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamond}_{W_{\mathcal{O}_E,e}(\ell)} \to \mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}(\mathcal{O}_E)} \mathrm{Spd}(W_{\mathcal{O}_E,e}(\ell))$$

is 2-isomorphic to the natural map of Lemma 3.1.6.

Remark 4.4.2. Under certain additional hypotheses on (G, X), it is conjectured in [HK19, Axiom A] that (for $\ell = \overline{\mathbb{F}}_p$) there should be a uniformization map $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,e}(\overline{\mathbb{F}}_p) \to \mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})(\overline{\mathbb{F}}_p)$. If $\Pi_{\mathcal{G}} \neq 1$, then such a map cannot upgrade to a uniformization map as in Theorem 4.4.1. Indeed, the natural map $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamond} \to \mathrm{Sht}_{\mathcal{G},\mu}$ factors through $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ by Remark 4.1.5, while the natural map $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,e} \to \mathrm{Sht}_{\mathcal{G},\mu}$ does not.

Proof. The proof of [GLX23, Corollary 6.3] goes through[15], with the following modification. In the notation of [GLX23, Section 3.4], we have an isomorphism, where the right hand side is the local Shimura variety of level $\mathcal{G}(\mathbb{Z}_p)$ over $\mathrm{Spd}(\breve{E})$ associated to $(G, b_x, \mu, \mathcal{G}(\mathbb{Z}_p))$,

$$\begin{aligned}\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e} &\times_{\mathrm{Spd}(W_{\mathcal{O}_E,e}(\ell))} \mathrm{Spd}(W_{\mathcal{O}_E,e}(\ell)[1/p]) \\ &\simeq \mathrm{Sht}_{G,b_x,\mu,\mathcal{G}(\mathbb{Z}_p)} \times_{\mathrm{Spd}(\breve{E})} \mathrm{Spd}(W_{\mathcal{O}_E,e}(\ell)[1/p])\end{aligned}$$

see [PR22, Theorem 4.5.1]. This means that we can follow the construction in [GLX23, Corollary 3.11] to construct a map

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \pi_0\left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}\right).$$

To show that this map is surjective, we use the commutative diagram

$$\begin{array}{ccc} G(\mathbb{Q}_p)/\mathcal{G}^\circ(\mathbb{Z}_p) & \longrightarrow & \pi_0\left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b_x,\mu,e}\right) \\ \downarrow & & \downarrow \\ G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) & \longrightarrow & \pi_0\left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}\right), \end{array}$$

and the surjectivity of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b_x,\mu,e} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}$ and $G(\mathbb{Q}_p)/\mathcal{G}^\circ(\mathbb{Z}_p) \to \pi_0\left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b_x,\mu,e}\right)$, see [GLX23, Corollary 3.11] for the latter. With this in mind, the rest of the proof of [GLX23, Corollary 6.3] goes through. □

Corollary 4.4.3. *For $z \in \mathscr{S}_{K_p^\circ}(\mathsf{G},\mathsf{X})(\ell)$ with image $x \in \mathscr{S}_{K_p}(\mathsf{G},\mathsf{X})(\ell)$, there is a uniformization map*

$$\Theta_{\mathcal{G}^\circ,z} : \mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b_x,\mu,e} \to \mathscr{S}_{K_p^\circ}(\mathsf{G},\mathsf{X})^\diamond_{W_{\mathcal{O}_E,e}(\ell)}$$

sending the base point x_0 to z, that restricts to an isomorphism

$$\Theta_{\mathcal{G}^\circ,x} \colon \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b_x,\mu,e}}_{/x_0} \xrightarrow{\cong} (\widehat{\mathscr{S}_{K_p^\circ}(\mathsf{G},\mathsf{X})_{W_{\mathcal{O}_E,e}(\ell)}}_{/z})^\diamond.$$

Proof. If we define Y (and $\Theta_{\mathcal{G}^\circ,z}$) as the fiber product

$$\begin{array}{ccc} Y & \xrightarrow{\Theta_{\mathcal{G}^\circ,z}} & \mathscr{S}_{K_p^\circ}(\mathsf{G},\mathsf{X})^\diamond_{W_{\mathcal{O}_E,e}(\ell)} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e} & \xrightarrow{\Theta_{\mathcal{G},x}} & \mathscr{S}_{K_p}(\mathsf{G},\mathsf{X})^\diamond_{W_{\mathcal{O}_E,e}(\ell)}, \end{array}$$

[15]We specifically mean the version of the proof linked in our bibliography, which differs from the Arxiv version at the time of writing.

then by concatenating fiber product squares (see the proof of Theorem 4.1.12 and Theorem 4.2.3) we get a fiber product diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Sht}_{\mathcal{G}^\circ,\mu} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e} & \longrightarrow & \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}. \end{array}$$

It follows from the proof of Theorem 3.3.5 and Lemma 3.1.6 that $Y \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}$ is isomorphic to $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^\circ,b_x,\mu,e} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,\delta=1,e}$, proving the corollary. □

Appendix A. On scheme-theoretic local model diagrams

In this appendix we flesh out the remark in [PR24, Section 4.9.2] that the local model diagrams of [KZ21, Theorem 5.1.10], [KP18, Theorem 4.2.7] and [KPZ24] give scheme-theoretic local model diagrams for integral models of Hodge-type Shimura varieties.

A.1. **Some rational p-adic Hodge theory.** Let X be a smooth rigid space over a finite extension E of $\mathbb{Q}_p$ with pro-étale site $X_{\text{proét}}$ as in [Sch13, Definition 3.9]. We will consider the period sheaves $\mathbb{B}_{\text{dR}}^+, \mathbb{B}_{\text{dR}}$ and $\mathcal{O}\mathbb{B}_{\text{dR}}$, see [Sch13, Definition 6.1, Definition 6.8]. Let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system of rank n on $X_{\text{proét}}$. Associated to $\mathbb{L}$ is a filtered vector bundle with integrable connection $D_{\text{dR}}(\mathbb{L}) = (\mathcal{E}, \text{Fil}^\bullet, \nabla)$ on $X_{\text{ét}}$ satisfying Griffiths transversality, see [LZ17, Theorem 3.9]. We have two $\mathbb{B}_{\text{dR}}^+$-lattices on $X_{\text{proét}}$:

$$\mathbb{M} := \mathbb{L} \otimes_{\underline{\mathbb{Z}_p}} \mathbb{B}_{\text{dR}}^+, \quad \text{and} \quad \mathbb{M}_0 := (D_{\text{dR}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\text{dR}}^+)^{\nabla=0}.$$

Here for the construction of $\mathbb{M}_0$, we take flat sections for the induced connection $\nabla = \nabla_{D_{\text{dR}}(\mathbb{L})} \otimes \text{id} + \text{id} \otimes \nabla_{\mathcal{O}\mathbb{B}_{\text{dR}}^+}$. Also, by [PR24, Definition 2.6.4, Proposition 2.6.3, Proposition 2.5.1], there is an induced shtuka $\mathscr{V}_{\mathbb{L}}$ of rank n on $X^\lozenge$.

Let $S = \text{Spa}(R, R^+)$ be an affinoid perfectoid space of characteristic p together with a map $f : S \to X^\lozenge$ corresponding to an untilt $S^\sharp$ and a map $f : S^\sharp \to X$. Note that by construction of $\mathscr{V}_{\mathbb{L}}$, the completion of $\mathscr{V}_{\mathbb{L}}$ (resp. $\text{Frob}_S^* \mathscr{V}_{\mathbb{L}}$) along $S^\sharp$ is canonically identified with $f^*\mathbb{M}$ (resp. $f^*\mathbb{M}_0$), where these pullbacks are defined as in the proof of [PR24, Proposition 2.6.3]. We equip $\text{Frob}_S^* \mathscr{V}_{\mathbb{L}}\big|_{S^\sharp}$ with a decreasing filtration such that (the Tate twist can be ignored)

$$\text{Fil}^{-i}(\text{Frob}_S^* \mathscr{V}_{\mathbb{L}}\big|_{S^\sharp}) := \mathbb{M} \cap \text{Fil}^i(\mathbb{M}_0)/\mathbb{M} \cap \text{Fil}^{i+1}(\mathbb{M}_0)(-i).$$

Lemma A.1.1. *There is a natural isomorphism between filtered vector bundles $D_{\text{dR}}(\mathbb{L})_{S^\sharp}$ and $\text{Frob}_S^* \mathscr{V}_{\mathbb{L}}\big|_{S^\sharp}$.*

Proof. The underlying vector bundle of $D_{\text{dR}}(\mathbb{L})|_{X_{\text{proét}}}$ can be recovered from $\mathbb{M}_0$ by taking 0th graded piece, see the discussion after the proof of [Sch13, Lemma 7.7]. Its filtration can be recovered from the relative position of $\mathbb{M}$ and $\mathbb{M}_0$, as explained in [Sch13, Proposition 7.9]. Since $\text{Frob}_S^* \mathscr{V}_{\mathbb{L}}\big|_{S^\sharp} = \text{gr}^0(\mathbb{M}_0)$, by comparing with the formula in [Sch13, Proposition 7.9], we see that it agrees with $D_{\text{dR}}(\mathbb{L})|_{S^\sharp}$ as filtered vector bundles. This identification is moreover natural in S. □

Note that Lemma A.1.1 gives us a 2-commutative diagram of tensor functors (discarding the connection on $D_{\text{dR}}(-)$)

$$\begin{array}{ccc} \{\text{de Rham } \mathbb{Z}_p\text{-local systems on } X_{\text{proét}}\} & \xrightarrow{\text{PR}} & \{\text{Shtukas on } X^\lozenge\} \\ \big\downarrow{\scriptstyle D_{\text{dR}}} & & \big\downarrow \\ \{\text{Filtered vector bundles on } X_{\text{ét}}\} & \longrightarrow & \{\text{Filtered vector bundles on } X^\lozenge\}. \end{array}$$

Here PR is the (exact) tensor functor of [PR24, Definition 2.6.4], and the right vertical arrow takes a shtuka $\mathscr{V}$ on $X^\diamondsuit$ to $\mathrm{Frob}_S^* \mathscr{V}_{\mathbb{L}}\big|_{S^\sharp}$ as in Lemma A.1.1.

A.1.2. Now suppose that $X = Z^{\mathrm{an}}$ for a smooth E-scheme Z, and that there is an abelian scheme $\pi : A \to Z$ of relative dimension g such that $\mathbb{L} \coloneqq R^1\pi_{*,\mathrm{pro\acute{e}t}}\underline{\mathbb{Z}_p}$. Then as explained in [PR24, Example 2.6.2],

$$D_{\mathrm{dR}}(\mathbb{L}) \simeq (\mathcal{H}^1_{\mathrm{dR}}(A/X), \mathrm{Fil}^\bullet_{\mathrm{Hdg}}),$$

where $\mathcal{H}^1_{\mathrm{dR}}(A/X)$ denotes the first relative de Rham cohomology of π, equipped with its Hodge filtration $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}$. To be precise, there is a natural surjective map of vector bundles $\mathcal{H}_1^{\mathrm{dR}}(A/X) \to \mathrm{Lie}(A^\vee)$ with kernel $\mathrm{Fil}^1_{\mathrm{Hdg}}$. Note that it follows from [CS17, Proposition 2.2.3] that $\mathbb{M}_0 \subset \mathbb{M}$. We let recall the element $\xi \in \mathbb{B}^+_{\mathrm{dR}}$ generating $\ker\theta$, see [Sch13, Section 6].

Lemma A.1.3. *We can identify $\xi\mathbb{M} \subset \mathbb{M}_0$ with the kernel of the map*

$$\mathbb{M}_0 \to H^1_{\mathrm{dR}}(A/X) \to \mathrm{Lie}(A^\vee)$$

Proof. Note that the Hodge filtration on $H^1_{\mathrm{dR}}(A/X)$ only has two jumps. The lemma now follows from the explicit formula of the filtration above. □

Let $\Lambda = \mathbb{Z}_p{}^{\oplus 2g}$ and let $P_\Lambda = \underline{\mathrm{Isom}}_X(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X, \mathcal{H}^1_{\mathrm{dR}}(A/X))$ be the frame bundle of $\mathcal{H}^1_{\mathrm{dR}}(A/X)$. Let $\mathrm{Gr}_{g,\Lambda}$ be the Grassmannian of g-dimensional quotients of Λ considered as scheme over $\mathbb{Z}_p$. Then there is a map of adic spaces

$$\pi_{\mathrm{dR}} : P_\Lambda \to \mathrm{Gr}^{\mathrm{an}}_{g,\Lambda,E}$$

defined using the natural quotient map $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X \cong \mathcal{H}^1_{\mathrm{dR}}(A/X) \to \mathrm{Lie}(A^\vee)$.

A.1.4. We adopt the notation from A.1.2 above. Let $\mathscr{V}$ be the vector bundle shtuka induced from $\mathbb{L}$. Then by Lemma A.1.3, $\mathscr{V}$ is minuscule of height $2g$ and dimension g in the sense of [PR24, Definition 2.2.2]. By [PR24, Lemma 2.4.4], we can think of it as a $\mathrm{GL}(\Lambda)$-shtuka bounded by the cocharacter $\mu_g = (1^{(g)}, 0^{(g)})$. Since $\mathrm{GL}(\Lambda)$ is a reductive group, we may identify the local model $\mathbb{M}_{\mathrm{GL}(\Lambda),\mu_g}$ with the flag variety $\mathrm{Gr}_{g,\Lambda}$ of g-dimensional quotients of Λ as $\mathcal{O}_E$-schemes (see [AGLR22, Example 4.12] and [SW20, Proposition 19.4.2]). We consider the diamond associated to the local model $\mathbb{M}^\diamondsuit_{\mathrm{GL}(\Lambda),\mu_g}$ as a closed subfunctor of the Beilinson–Drinfeld affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}(\Lambda)}$ for $\mathrm{GL}(\Lambda)$.

On the generic fiber (base changed to E), the isomorphism $\mathbb{M}^\diamondsuit_{\mathrm{GL}(\Lambda),\mu_g,E} \xrightarrow{\sim} \mathrm{Gr}^\diamondsuit_{g,\Lambda,E}$ is induced by the Białynicki-Birula map, see [SW20, Proposition 19.4.2].

A.1.5. Let the notation be as in Section A.1.2. We can define a $\mathrm{GL}(\Lambda)^\diamondsuit$-torsor of trivializations $\mathcal{P}_\Lambda$ over $X^\diamondsuit$ via

$$(S \to X^\diamondsuit) \mapsto \mathrm{Isom}_{\mathcal{O}_{S^\sharp}}(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{S^\sharp}, \phi^*\mathscr{V}\big|_{S^\sharp}).$$

Lemma A.1.1 implies that there is a canonical isomorphism of $\mathrm{GL}(\Lambda)^\diamondsuit$-torsors over $X^\diamondsuit$

$$P_\Lambda^\diamondsuit \xrightarrow{\sim} \mathcal{P}_\Lambda.$$

Applying the construction of [PR24, Section 4.9.1], we get a diagram

$$X^\diamondsuit \leftarrow \mathcal{P}_\Lambda \rightarrow \mathbb{M}^\diamondsuit_{\mathrm{GL}(\Lambda),\mu_g,E}.$$

The right arrow is $\mathrm{GL}(\Lambda)^\diamondsuit$-equivariant, and following the notation in *loc. cit.*, it is constructed by (locally on S) lifting a section of $\mathcal{P}_\Lambda$ to an isomorphism over $\widehat{S^\sharp} := \mathrm{Spec}(\hat{\mathcal{O}}_{S\dot{\times}\mathrm{Spa}\,\mathbb{Z}_p,S^\sharp})$, and then send it to the triple

$$(S^\sharp, \mathscr{V}, \alpha : \mathscr{V}|_{\widehat{S^\sharp}\setminus S^\sharp} \xrightarrow[\sim]{\phi_{\mathscr{V}}^{-1}} \mathrm{Frob}^*\mathscr{V}|_{\widehat{S^\sharp}\setminus S^\sharp} \simeq \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{S^\sharp}) \in \mathrm{Gr}_{\mathrm{GL}(\Lambda),E}(S).$$

Its image lies in the minuscule Schubert cell $\mathbb{M}^\diamondsuit_{\mathrm{GL}(\Lambda),\mu_g,E} = \mathrm{Gr}_{\mathrm{GL}(\Lambda),\mu_g,E}$. By Lemma A.1.1, we have the following compatibility.

Proposition A.1.6. *The diagram below commutes, where the vertical isomorphisms are the ones from Sections A.1.5 and A.1.4.*

$$\begin{array}{ccccc} X^\diamondsuit & \longleftarrow & \mathcal{P}_\Lambda & \longrightarrow & \mathbb{M}^\diamondsuit_{\mathrm{GL}(\Lambda),\mu_g,E} \\ \| & & \downarrow{\sim} & & \downarrow{\sim} \\ X^\diamondsuit & \longleftarrow & P_\Lambda^\diamondsuit & \longrightarrow & \mathrm{Gr}^\diamondsuit_{g,\Lambda,E} \end{array}$$

A.1.7. Now let (G,X) be a Shimura datum with Hodge cocharacter μ and reflex field E satisfying (4.0.2). Let $v|p$ be a place of E, $E := \mathsf{E}_v$, and $\mathcal{G}$ be a parahoric model of G over $\mathbb{Q}_p$. Let $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $K = K^pK_p$ for $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ a neat compact open subgroup. We now specialize the previous section to the situation that $X = \mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\mathrm{an}}$. Then there is a pro-étale $\mathcal{G}(\mathbb{Z}_p)$-torsor $\mathbb{P} \to X$ which is de Rham in the sense of [PR24, Definition 2.6.5], see [PR24, Section 4.1]. This gives us an exact tensor functor

$$\mathbb{L}_p : \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G} \to \{\text{de Rham } \mathbb{Z}_p\text{-local systems on } X_{\text{proét}}\}$$
$$W \mapsto \mathbb{P} \times^{\mathcal{G}(\mathbb{Z}_p)} \underline{W}.$$

The composition $D_{\mathrm{dR}} \circ \mathbb{L}_p$ defines a G^{an}-torsor P on $X_{\text{ét}}$ via the Tannakian formalism. It thus follows from Lemma A.1.1 that the $\mathcal{G}^\diamondsuit$-torsor $P^\diamondsuit$ on $X^\diamondsuit$ is naturally isomorphic to the $\mathcal{G}^\diamondsuit$-torsor $\mathcal{P}_{\mathrm{PR}}$ induced by the $\mathcal{G}$-shtuka over $X^\diamondsuit$ coming from $\mathbb{L}_p$. We note that since the filtered vector bundles in the essential image of $D_{\mathrm{dR}} \circ \mathbb{L}_p$ are equipped with a decreasing filtration of type μ, the torsor P has a canonical reduction of structure group to the standard parabolic attached to μ[16], see [LZ17, Remark 4.1(i)]. Therefore it admits a map $P \to \mathcal{F}\ell_{G,\mu} := (G/P_\mu^{\mathrm{std}})^{\mathrm{an}}_E$. On the other hand, similar to what we have explained in Section A.1.5, the construction in

[16]We follow the convention in [CS17, Section 2.1] for parabolics attached to cocharacters.

[PR24, Section 4.9.1] gives a map $\mathcal{P}_{\mathrm{PR}} \to \mathrm{Gr}_{G,E}$ with image in the Schubert cell $\mathrm{Gr}_{G,\mu,E} = \mathbb{M}^{\Diamond}_{\mathcal{G},\mu,E}$. Moreover, the following diagram is commutative,

$$\begin{array}{ccccc} X^{\Diamond} & \longleftarrow & \mathcal{P}_{\mathrm{PR}} & \longrightarrow & \mathbb{M}^{\Diamond}_{\mathcal{G},\mu,E} \\ \| & & \downarrow\wr & & \downarrow\wr \\ X^{\Diamond} & \longleftarrow & P^{\Diamond} & \longrightarrow & \mathcal{F}\ell_{G,\mu}, \end{array}$$

where the rightmost vertical arrow is induced by the Białynicki-Birula map. Note that there is also an exact tensor functor (the canonical construction)

$$\mathcal{L} : \mathrm{Rep}_{\mathbb{Z}_p} \mathcal{G} \to \{\text{Filtered vector bundles on } X\},$$

see [DLLZ23, Proposition 5.2.10]. It follows from [DLLZ23, Theorem 5.3.1] that there is a natural isomorphism of tensor functors

$$\mathcal{L} \xrightarrow{\sim} D_{\mathrm{dR}} \circ \mathbb{L}_p.$$

Thus the $\mathcal{G}^{\Diamond}$-torsor $\mathcal{P}_{\mathrm{dR}}$ on $X^{\Diamond}$ corresponding to $\mathcal{L}$ via the Tannakian formalism, is naturally isomorphic to the $\mathcal{G}^{\Diamond}$-torsor P_{PR} induced by the $\mathcal{G}$-shtuka over $X^{\Diamond}$ coming from $\mathbb{L}_p$. This isomorphism is moreover compatible with filtrations and thus with the map to $\mathbb{M}^{\Diamond}_{\mathcal{G},\mu,E}$.

A.2. **Some integral p-adic Hodge theory.** Let $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ be an untilt of an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ in characteristic p and let $\mathfrak{A}$ be the completion of an abelian scheme over $R^{\sharp+}$ with associated p-divisible group $Y = \mathfrak{A}[p^\infty]$. By [SW20, Theorem 17.5.2], we can associate to Y a finite free $W(R^+)$-module $M(Y)$ equipped with an isomorphism

$$\phi_M : \phi^* M(Y)[1/\phi(\xi)] \xrightarrow{\sim} M(Y)[1/\phi(\xi)]$$

such that

$$M(Y) \subset \phi_M\left(\phi^* M(Y)\right) \subset \tfrac{1}{\phi(\xi)} M(Y).$$

Here ξ is a generator of the kernel of $W(R^+) \to R^{\sharp+}$. Let $M(Y)^*$ denote the $W(R^+)$-linear dual of $M(Y)$, which we will equip with the isomorphism ϕ_{M^*} given by the inverse of the $W(R^+)$-linear dual of ϕ_M. This is the (contravariant) prismatic Dieudonné module of Y and it satisfies

$$\xi M(Y)^* \subset \phi_{M^*}(\phi^* M(Y)^*) \subset M(Y)^*.$$

By restriction along $S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p \to \mathrm{Spec}\, W(R^+)$, it gives rise to a minuscule vector bundle shtuka with one leg at $S^\sharp$.

Lemma A.2.1. *There is a canonical isomorphism*

$$M(Y)^* \otimes_{W(R^+)} R^{\sharp+} \xrightarrow{\sim} H^1_{\mathrm{dR}}(A/R^{\sharp+})$$

compatible with base change.

Proof. We may identify $M(Y)^*$ with the ϕ-pullback of the relative prismatic cohomology of $\mathfrak{A}$ using [ALB23, Corollary 4.63, Proposition 4.49].[17] The comparison isomorphism now follows from [BS22, Theorem 1.8.(3)]. ☐

A.2.2. For a characteristic zero untilt $R^\sharp$, we want to compare the isomorphism of Lemma A.2.1 with the isomorphism of Lemma A.1.1. We will do this under the assumption that $\mathfrak{A}$ is the pullback of a formal abelian scheme $f : \mathfrak{B} \to \mathfrak{X}$ over a smooth formal scheme $\mathfrak{X}/\mathcal{O}_K$ for some discrete valued field $K/\mathbb{Q}_p$ (which will be the case in our situation since the Siegel modular variety is smooth). Denote the special fiber of $\mathfrak{X}$ by $\mathfrak{X}_s$ and the rigid generic fiber of $\mathfrak{X}$ by X, and similarly for $\mathfrak{B}$.

Note that the F-isocrystal $\mathcal{E}$ on $\mathfrak{X}_s$ obtained by the contravariant Dieudonné crystal of $\mathfrak{B}_s$ is associated to the vector bundle with flat connection $(E, \nabla) := (R^1 f_{\mathrm{dR},*}\mathcal{O}_B, \nabla_{\mathrm{GM}})$ (∇_{GM} denotes the Gauss-Manin connection) on X, in the sense of [GR24, Proposition 2.17]. Then the proof of Proposition 2.36.(i) in *loc. cit.* shows that $E \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ equipped with the product connection is isomorphic to $(\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+, \mathrm{id} \otimes \nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+})$. In particular, we have a natural identification of the horizontal sections

$$\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) = (\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)^{\nabla=0} \cong (E \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)^{\nabla=0} =: \mathbb{M}_0.$$

On the other hand, under the prismatic–crystalline comparison [BS22, Theorem 1.8(1)], we have that

$$\begin{aligned} \mathbb{B}_{\mathrm{dR}}^+(\mathcal{E})(S^\sharp) &= \mathbb{A}_{\mathrm{crys}}(\mathcal{E}) \otimes_{\mathbb{A}_{\mathrm{crys}}} \mathbb{B}_{\mathrm{dR}}^+(S^\sharp) \\ &= R^1 f_{\mathrm{crys},*}\mathcal{O} \otimes_{\mathbb{A}_{\mathrm{crys}}} \mathbb{B}_{\mathrm{dR}}^+(S^\sharp) \\ &\xrightarrow{\sim} \phi^* H^1_{\mathbb{\Delta}}(\mathfrak{A}/\mathbb{\Delta}_{R^{\sharp+}}) \otimes_{W(R^+)} \mathbb{B}_{\mathrm{dR}}^+(S^\sharp) \\ &= M(Y)^* \otimes_{W(R^+)} B_{\mathrm{dR}}^+(R^\sharp). \end{aligned} \tag{A.2.1}$$

Here $\mathbb{\Delta}_{R^{\sharp+}}$ denotes the perfect prism $(W(R^+), \ker\theta = (\xi))$. Note that the definition of the de Rham period sheaves in [GR24] differs from ours by a Frobenius twist, see Definition 2.3, Warning 2.4 in *loc. cit.*, but their arguments work verbatim. We conclude that the following diagram commutes (cf. [IKY23, Lemma 2.18])

$$\begin{array}{ccc} M(Y)^* & \longrightarrow & M(Y)^* \otimes_{W(R^+)} B_{\mathrm{dR}}^+(R^\sharp) \\ \downarrow & & \downarrow\wr \\ H^1_{\mathrm{dR}}(A/R^\sharp) & \longleftarrow & \mathbb{M}_0(S^\sharp). \end{array}$$

Here the left vertical map is the map from Lemma A.2.1 composed with inverting p and the bottom horizontal map is the map from Lemma A.1.3. The right vertical map is the comparison isomorphism from equation (A.2.1).

[17] In [ALB23, Proposition 4.49], the Frobenius twist is hidden in the notation $\tilde{\xi} = \phi(\xi)$.

A.2.3. The discussion in Section A.2.2 above implies an integral version of the result in Section A.1.5: Following the notation in Section A.1.2. Suppose $A \to X$ is the rigid generic fiber of a family of formal abelian schemes $\mathfrak{A} \to \mathfrak{X}$, for some smooth formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\mathcal{O}_E$. By descending the relative (contravariant) prismatic Dieudonné crystal of the pullback of $\mathfrak{A}$ to integral perfectoids over $\mathfrak{X}$, one has a vector bundle shtuka $\mathscr{V}$ over $\mathfrak{X}^\lozenge$. We can define a $\mathrm{GL}(\Lambda)^\lozenge$-torsor of trivializations $\mathcal{P}_\Lambda$ over $\mathfrak{X}^\lozenge$ via

$$(S \to \mathfrak{X}^\lozenge) \mapsto \mathrm{Isom}_{\mathcal{O}_{S^\sharp}}(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{S^\sharp}, \phi^*\mathscr{V}\big|_{S^\sharp}).$$

Similarly, one can consider the frame bundle P_Λ of $\mathcal{H}^1_{\mathrm{dR}}(\mathfrak{A}/\mathfrak{X})$. Lemma A.2.1 implies that there is a canonical isomorphism $P_\Lambda^\diamond \times^{\mathrm{GL}_\Lambda^\diamond} \mathrm{GL}_\Lambda^\lozenge \xrightarrow{\sim} \mathcal{P}_\Lambda$ of $\mathrm{GL}(\Lambda)^\lozenge$-torsor over $\mathfrak{X}^\lozenge$.

Repeating the procedure in Section A.1.5 we get a commutative diagram (note that we get a $\mathrm{GL}_\Lambda^\diamond$-torsor over $\mathfrak{X}^\lozenge$ because the frame bundle is a torsor for the p-adic completion of GL_Λ whose associated big diamond ($\lozenge$) gives $\mathrm{GL}_\Lambda^\diamond$)

$$\begin{array}{ccccc} \mathfrak{X}^\lozenge & \longleftarrow & \mathcal{P}_\Lambda & \longrightarrow & \mathbb{M}^\lozenge_{\mathrm{GL}(\Lambda),\mu_g} \\ \| & & \downarrow\wr & & \downarrow\wr \\ \mathfrak{X}^\lozenge & \longleftarrow & P_\Lambda^\diamond \times^{\mathrm{GL}_\Lambda^\diamond} \mathrm{GL}_\Lambda^\lozenge & \longrightarrow & \mathrm{Gr}^\lozenge_{g,\Lambda}. \end{array}$$

By the discussion in Section A.2.2, it is compatible with the one in Proposition A.1.6 when passing to the generic fiber.

A.3. **Shimura varieties of Hodge type.** We follow the notation in the proof of Theorem 4.2.3. In particular, we have $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ with $\mathcal{G}$ a stabilizer Bruhat–Tits group scheme, a Hodge embedding $\iota : (\mathsf{G}, \mathsf{X}) \to (\mathsf{G}_V, \mathsf{H}_V)$. We may moreover take a $\mathbb{Z}_p$-lattice $\Lambda \subset V_{\mathbb{Q}_p}$ on which ψ is $\mathbb{Z}_p$-valued, such that $\mathcal{G}(\breve{\mathbb{Z}}_p)$ is the stabilizer in $G(\breve{\mathbb{Q}}_p)$ of $V_p \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$. Assume moreover that the Hodge embedding ι is good for $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ with respect to Λ in the sense of [KZ21, Definition 3.3.15], which means that the natural maps

$$\mathcal{G} \to \mathrm{GL}(\Lambda) \quad \text{and} \quad \mathbb{M}_{\mathcal{G},\mu} \to \mathrm{Gr}_{g,\Lambda}$$

are closed immersions.[18]

Remark A.3.1. If p is coprime to $2 \cdot \pi_1(G^{\mathrm{der}})$ and G is acceptable ([KZ21, Definition 3.3.2]) and R-smooth ([DY24, Definition 2.10]), then a good Hodge embedding always exists, see [KZ21, Lemma 5.1.3]. By Zarhin's trick, see [Kis17, Section 1.3.3], we may moreover replace Λ by $\Lambda^{\oplus 4} \oplus \Lambda^{\vee,\oplus 4}$ to assume that Λ is self dual. The resulting Hodge embedding is then still good with respect to Λ.

[18] The local models used by [KZ21] agree with ours because theirs also satisfy the Scholze–Weinstein conjecture (which means that they have the correct associated v-sheaf), see [KZ21, Proposition 3.3.10].

A.3.2. Let $P_{\Lambda^\vee} \to \mathscr{S}_K(\mathsf{G},\mathsf{X})$ be the $\mathrm{GL}(\Lambda^\vee)$-torsor parametrizing trivialisations of the de Rham cohomology of the universal abelian variety (up to prime-to-p isogeny) coming from ι. Then there is a morphism $P_{\Lambda^\vee} \to \mathrm{Gr}_{g,\Lambda^\vee,\mathcal{O}_E}$ as in Section A.1.2. By the proof of [KZ21, Theorem 5.1.10], there is a $\mathcal{G}$-torsor $P \to \mathscr{S}_K(\mathsf{G},\mathsf{X})$ together with a $\mathcal{G}$-equivariant map $P \to P_{\Lambda^\vee}$ such that the composition

$$P \to P_{\Lambda^\vee} \to \mathrm{Gr}_{g,\Lambda^\vee,\mathcal{O}_E}$$

factors through $\mathbb{M}_{\mathcal{G},\mu}$ via a smooth map. To compare with the constructions in [PR24], which considers the $\mathrm{GL}(\Lambda)$-torsor of isomorphisms from the de Rham homology to Λ (rather than cohomology to $\Lambda^\vee$), we push out along the natural isomorphism $\mathrm{GL}(\Lambda^\vee) \to \mathrm{GL}(\Lambda)$ and obtain a $\mathrm{GL}(\Lambda)$-torsor P_Λ with a diagram

$$\mathscr{S}_K(\mathsf{G},\mathsf{X}) \leftarrow P_\Lambda \to \mathrm{Gr}_{g,\Lambda,\mathcal{O}_E}.$$

As before we have a $\mathcal{G}$-torsor $P \subset P_\Lambda$ such that the above diagram restricts to

$$\mathscr{S}_K(\mathsf{G},\mathsf{X}) \leftarrow P \to \mathbb{M}_{\mathcal{G},\mu},$$

where the left arrow is a $\mathcal{G}$-torsor and the right arrow is smooth and $\mathcal{G}$-equivariant. It moreover follows from the construction that its generic fiber comes from the canonical model of the standard principal bundle, see the discussion in the proof of [CS17, Lemma 2.3.5]. Let us write

$$\pi_{\mathrm{dR},\mathcal{G}} \colon \mathscr{S}_K(\mathsf{G},\mathsf{X}) \to [\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}]$$

for the induced smooth morphism of algebraic stacks.

Theorem A.3.3. *If the Hodge embedding ι is good for $(\mathsf{G},\mathsf{X},\mathcal{G})$ with respect to a self-dual lattice Λ, then the morphism $\pi_{\mathrm{dR},\mathcal{G}}$ is a scheme-theoretic local model diagram.*

For the proof of Theorem A.3.3, we will need the following two lemmas.

Lemma A.3.4. *Let $\mathscr{Y}$ be a v-sheaf which is separated over $\mathrm{Spd}(\mathbb{Z}_p)$. For any normal scheme X which is flat, separated and of finite-type over $\mathbb{Z}_p$, the natural restriction map*

$$\mathrm{Hom}_{\mathrm{Spd}(\mathbb{Z}_p)}(X^{\Diamond/}, \mathscr{Y}) \to \mathrm{Hom}_{\mathrm{Spd}(\mathbb{Q}_p)}(X^{\Diamond}_{\mathbb{Q}_p}, \mathscr{Y}_{\mathbb{Q}_p})$$

is injective.

Proof. This follows from the density of $|X^{\Diamond}_{\mathbb{Q}_p}| \subset |X^{\Diamond/}|$, which in turn follows from the density of $|(X^{\diamond})_{\mathbb{Q}_p}| \subset |X^{\diamond}|$ (see [AGLR22, Lemma 2.17]). ☐

Lemma A.3.5. *The quotient v-sheaf $\mathrm{GL}(\Lambda)^{\Diamond}/\mathcal{G}^{\Diamond}$ is separated over $\mathrm{Spd}(\mathbb{Z}_p)$.*

Proof. By [HdS21, Lemma 6.17], there is a finite free representation W of $\mathrm{GL}(\Lambda)$ and a free rank-one saturated $\mathbb{Z}_p$-submodule $L \subset W$ for which $\mathcal{G}$ is the scheme-theoretic stabilizer of L inside of $\mathrm{GL}(\Lambda)$. It follows that there is a morphism of v-sheaves

$$\mathrm{GL}(\Lambda)^{\Diamond}/\mathcal{G}^{\Diamond} \to \mathbb{P}(W)^{\Diamond}, \tag{A.3.1}$$

defined at the level of presheaves by $[g] \mapsto g \cdot L$. We claim this is a monomorphism. Indeed, suppose S is a perfectoid space in characteristic p, and that $a, b \colon S \to \mathrm{GL}(\Lambda)^{\Diamond}/\mathcal{G}^{\Diamond}$ are two morphisms which agree after the composition to $\mathbb{P}(W)^{\Diamond}$. After replacing S by a v-cover, we may assume a and b factor through morphisms $\tilde{a}$, $\tilde{b}\colon S \to \mathrm{GL}(\Lambda)^{\lozenge}$. Since a and b agree after the composition to $\mathbb{P}(W)^{\lozenge}$, and $\mathcal{G}$ is the stabilizer of L, it follows that $\tilde{a} \cdot \tilde{b}^{-1}$ factors through $\mathcal{G}^{\lozenge}$; thus $a = b$.

Since (A.3.1) is a monomorphism, its diagonal is an isomorphism, and therefore (A.3.1) is separated. Now $\mathbb{P}(W)$ is proper over $\mathrm{Spec}(\mathbb{Z}_p)$, so $\mathbb{P}(W)^{\lozenge} = \mathbb{P}(W)^{\diamond}$, and hence $\mathbb{P}(W)^{\lozenge} \to \mathrm{Spd}(\mathbb{Z}_p)$ is separated by [Gle20, Proposition 4.17]. The result follows. □

Proof of Theorem A.3.3. We start by identifying

$$P^{\lozenge/} \times^{\mathcal{G}^{\lozenge/}} \mathcal{G}^{\lozenge}$$

with the $\mathcal{G}^{\lozenge}$-torsor coming from the map $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G},\mu}$. We first check that this holds after composing with $\mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Sht}_{\mathrm{GL}(\Lambda),\mu_g}$ and pushing out via

$$\mathcal{G}^{\lozenge} \to \mathrm{GL}(\Lambda)^{\lozenge}.$$

The latter result is true over $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\lozenge}$ by Proposition A.1.6. Moreover, by Sections A.2.2 and A.2.3, it is true over $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\diamond}$[19], such that induced isomorphisms agree on $(\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\diamond})_E$, so they glue to an isomorphism over $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\lozenge/}$.

Next, we check that the induced $\mathcal{G}^{\lozenge}$-torsors agree: after trivializing the induced $\mathrm{GL}(\Lambda)$-torsor, which we may do Zariski locally on $\mathscr{S}_K(\mathsf{G},\mathsf{X})$, we are trying to show the equality of two morphisms $\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\lozenge/} \to \mathrm{GL}(\Lambda)^{\lozenge}/\mathcal{G}^{\lozenge}$. By Lemma A.3.5 and Lemma A.3.4, it suffices to check this after base change to the generic fiber, where the result follows from the discussion in Section A.1.7.

Finally, we check that, under this identification, the maps to the local model agree. But $\mathbb{M}_{\mathcal{G},\mu}$ is projective and hence proper over $\mathrm{Spec}(\mathcal{O}_E)$, so $\mathbb{M}_{\mathcal{G},\mu}^{\lozenge} \to \mathrm{Spd}(\mathcal{O}_E)$ is separated by [Gle20, Proposition 4.17]. By another application of Lemma A.3.4, it suffices to check the morphisms agree after base change to the generic fiber, where the result follows from the discussion in Section A.1.7. Together with the results implied by [KZ21, Theorem 5.1.10] discussed above, this concludes the proof that $\pi_{\mathrm{dR},\mathcal{G}}$ is a scheme-theoretic local model diagram. □

[19] The discussion there assumes that our (formal) abelian scheme comes via pullback from a (formal) abelian scheme over a smooth (formal scheme). This assumptions holds here since Λ is self dual and thus the integral model of the Shimura variety for $(\mathsf{G}_V, \mathsf{H}_V)$ is smooth.

References

[AGLR22] Johannes Anschütz, Ian Gleason, João Lourenço, and Timo Richarz, *On the p-adic theory of local models*, arXiv 2201.01234, January 2022. 3, 8, 14, 19, 28, 33, 40, 45

[ALB23] Johannes Anschütz and Arthur-César Le Bras, *Prismatic Dieudonné theory*, Forum Math. Pi **11** (2023), Paper No. e2, 92. MR 4530092 43

[ALY22] Piotr Achinger, Marcin Lara, and Alex Youcis, *Specialization for the pro-étale fundamental group*, Compos. Math. **158** (2022), no. 8, 1713–1745. MR 4490930 14, 15

[Ans22] Johannes Anschütz, *Extending torsors on the punctured* $\mathrm{Spec}(A_{\mathrm{inf}})$, J. Reine Angew. Math. **783** (2022), 227–268. MR 4373246 29

[Ans23] ———, *G-bundles on the absolute Fargues-Fontaine curve*, Acta Arith. **207** (2023), no. 4, 351–363. MR 4591255 7, 17

[Bor98] Mikhail Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50. MR 1401491 10

[BS22] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) **196** (2022), no. 3, 1135 – 1275. 43

[BT84] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316 1

[CS17] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766. MR 3702677 40, 41, 45

[Dan22] Patrick Daniels, *Canonical integral models for Shimura varieties defined by tori*, arXiv 2207.09513, July 2022. 2, 28

[DvanHKZ24] Patrick Daniels, Pol van Hoften, Dongryul Kim, and Mingjia Zhang, *Igusa stacks for Shimura varieties of Hodge type*, 2024, In preparation. 4

[DLLZ23] Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu, *Logarithmic Riemann-Hilbert correspondences for rigid varieties*, J. Amer. Math. Soc. **36** (2023), no. 2, 483–562. MR 4536903 42

[DY24] Patrick Daniels and Alex Youcis, *Canonical Integral Models of Shimura Varieties of Abelian Type*, arXiv 2402.05727, February 2024. 2, 3, 35, 44

[FS21] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv 2102.13459, 2021. 5, 6, 7, 14, 18

[GI23] Ian Gleason and Alexander B. Ivanov, *Meromorphic vector bundles on the Fargues–Fontaine curve*, arXiv 2307.00887, July 2023. 16, 24

[Gle20] Ian Gleason, *Specialization maps for Scholze's category of diamonds*, arXiv 2012.05483, 2020. 8, 13, 18, 25, 27, 46

[Gle21] Ian Gleason, *On the geometric connected components of moduli spaces of p-adic shtukas and local Shimura varieties*, arXiv 2107.03579, July 2021, p. arXiv:2107.03579. 16, 18

[GLX23] Ian Gleason, Dong Gyu Lim, and Yujie Xu, *The connected components of affine Deligne-Lusztig varieties*, `https://ianandreigf.github.io/Website/QsplitADLV.pdf`, January 2023. 3, 16, 37

[GR24] Haoyang Guo and Emanuel Reinecke, *A prismatic approach to crystalline local systems*, Inventiones Mathematicae **236** (2024), 1–164. 43

[HdS21] Phùng Hô Hai and João Pedro dos Santos, *On the structure of affine flat group schemes over discrete valuation rings, II*, Int. Math. Res. Not. IMRN (2021), no. 12, 9375–9424. MR 4276322 45

[He16] Xuhua He, *Kottwitz-Rapoport conjecture on unions of affine Deligne-Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 5, 1125–1141. 18, 20

[HK19] Paul Hamacher and Wansu Kim, *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann. **375** (2019), no. 3-4, 973–1044. MR 4023369 36

[vanH20] Pol van Hoften, *Mod p points on Shimura varieties of parahoric level (with an appendix by Rong Zhou)*, arXiv 2010.10496, October 2020. 3

[HR08] T. Haines and M. Rapoport, *On parahoric subgroups (Appendix to "Twisted loop groups and their affine flag varieties" by Pappas–Rapoport)*, 2008. 12

[Hub94] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513 – 551. 8

[Hub96] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, vol. 30, Springer Fachmedien Wiesbaden, 1996. 15

[IKY23] Naoki Imai, Hiroki Kato, and Alex Youcis, *The prismatic realization functor for Shimura varieties of abelian type*, arXiv 2310.08472, October 2023. 2, 43

[Ked20] Kiran S. Kedlaya, *Some ring-theoretic properties of* $\mathrm{A}_{\mathrm{inf}}$, p-adic hodge theory, Simons Symp., Springer, Cham, [2020] ©2020, pp. 129–141. MR 4359204 24

[Kim24] Dongryul Kim, *Descending finite projective modules from a Novikov ring*, arXiv 2402.17852, February 2024. 4, 14

[Kis10] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012. MR 2669706 1, 29

[Kis17] ———, mod p *points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914. MR 3630089 44

[KL15] Kiran S. Kedlaya and Ruochuan Liu, *Relative p-adic Hodge theory: foundations*, Astérisque (2015), no. 371, 239. MR 3379653 19

[KMPS22] Mark Kisin, Keerthi Madapusi Pera, and Sug Woo Shin, *Honda–Tate theory for Shimura varieties*, Duke Math. J. **171** (2022), no. 7, 1559–1614. MR 4484214 7, 32

[Kot97] Robert E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339. MR 1485921 10

[KP18] M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218. MR 3905466 3, 4, 5, 32, 33, 39

[KP23] Tasho Kaletha and Gopal Prasad, *Bruhat-Tits theory—a new approach*, New Mathematical Monographs, vol. 44, Cambridge University Press, Cambridge, 2023. MR 4520154 10, 11, 12, 21, 29

[KPZ24] Mark Kisin, George Pappas, and Rong Zhou, *Integral models of Shimura varieties with parahoric level structure II*, 2024, in preparation. 39

[KSZ21] Mark Kisin, Sug Woo Shin, and Yihang Zhu, *The stable trace formula for Shimura varieties of abelian type*, arXiv 2110.05381, October 2021. 26

[KZ21] Mark Kisin and Rong Zhou, *Independence of ℓ for Frobenius conjugacy classes attached to abelian varieties*, `https://www.dpmms.cam.ac.uk/~rz240/l-indep_v2.pdf`, March 2021. 3, 5, 33, 35, 39, 44, 45, 46

[LZ17] Ruochuan Liu and Xinwen Zhu, *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Invent. Math. **207** (2017), no. 1, 291–343. MR 3592758 39, 41

[Mil90] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 283–414. MR 1044823 33

[Mil92] James S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces, Univ. Montréal, Montreal, QC, 1992, pp. 151–253. MR 1155229 1

[Mil05] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 265–378. MR 2192012 2, 26

[Moo98] Ben Moonen, *Models of Shimura varieties in mixed characteristics*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 267–350. MR 1696489 1

[Pap23] Georgios Pappas, *On integral models of Shimura varieties*, Math. Ann. **385** (2023), no. 3-4, 2037–2097. MR 4566689 2

[PR22] Georgios Pappas and Michael Rapoport, *On integral local Shimura varieties*, arXiv 2204.02829, April 2022. 5, 10, 12, 13, 19, 20, 21, 22, 23, 24, 25, 27, 30, 31, 34, 37

[PR23] Georgios Pappas and Michael Rapoport, *On tamely ramified $\mathcal{G}$-bundles on curves.* 11

[PR24] ———, *p-adic shtukas and the theory of global and local Shimura varieties*, Camb. J. Math. **12** (2024), no. 1, 1–164. MR 4701491 1, 2, 3, 4, 5, 6, 8, 16, 17, 18, 19, 24, 27, 28, 29, 30, 32, 39, 40, 41, 42, 45

[Rap05] Michael Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque (2005), no. 298, 271–318, Automorphic forms. I. MR 2141705 1, 12

[Rap18] ———, *Accessible and weakly accessible period domains*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 4, 811–863, appendix to *On the p-adic cohomology of the Lubin–Tate tower* by Peter Scholze. MR 3861564 19

[Rod22] J. E. Rodríguez Camargo, *Locally analytic completed cohomology*, arXiv 2209.01057, August 2022, p. arXiv:2209.01057. 27

[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181. MR 1411570 7

[Sch13] Peter Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77. MR 3090230 39, 40

[Sch17] Peter Scholze, *Etale cohomology of diamonds*, arXiv 1709.07343, September 2017. 7, 14, 15, 18, 25

[Sta24] The Stacks project authors, *The stacks project*, `https://stacks.math.columbia.edu`, 2024. 14, 15

[SW13] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR 3272049 7

[SW20] ———, *Berkeley lectures on p-adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020. MR 4446467 2, 6, 7, 8, 15, 16, 17, 19, 35, 40, 42

[Tak24] Yuta Takaya, *Moduli spaces of level structures on mixed characteristic local shtukas*, arxiv 2402.07135, February 2024. 2

[Vie21] Eva Viehmann, *On Newton strata in the B_{dR}^+-Grassmannian*, arXiv 2101.07510, January 2021. 7

[Zha23] Mingjia Zhang, *A PEL-type Igusa Stack and the p-adic Geometry of Shimura Varieties*, arXiv 2309.05152, September 2023. 9, 17, 24, 25

[Zho20] Rong Zhou, *Mod p isogeny classes on Shimura varieties with parahoric level structure*, Duke Math. J. **169** (2020), no. 15, 2937–3031. 3

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